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Dominant and subdominant positive solutions to generalized Dickman equation

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ABSTRACT

The paper considers a generalized Dickman equation

$$t\dot{x}(t) = -\sum_{i=1}^{s} a_i x(t-\tau_i)$$

for $t \to \infty$ where $s \in \mathbb{N}$, $a_i > 0$, $\tau_i > 0$, $i = 1, \ldots, s$ and $\sum_{i=1}^{s} a_i = 1$. It is proved that there are two mutually disjoint sets of positive decreasing solutions such that, for every two solutions from different sets, the limit of their ratio for $t \to \infty$ equals 0 or ∞. The asymptotic behavior of such solutions is derived and a structure formula utilizing such solutions and describing all the solutions of a given equation is discussed. In addition, a criterion is proved giving sufficient conditions for initial functions to generate solutions falling into the first or the second set. Illustrative examples are given. Some open problems are suggested to be solved.

1. Introduction and preliminaries

For $t \to \infty$, the paper studies the asymptotic behavior of solutions to the equation

$$t\dot{x}(t) = -\sum_{i=1}^{s} a_i x(t-\tau_i) \tag{1}$$

where

$$t \ge t_0, \quad t_0 \in \mathbb{R}, \quad t_0 > r, \tag{2}$$

$0 < \tau_i \le r := \max\{\tau_1, \ldots, \tau_s\}$, $i = 1, \ldots, s$, $s \in \mathbb{N}$ and $a_i > 0$, $i = 1, \ldots, s$. Moreover, in a majority of statements, we assume

$$\sum_{i=1}^{s} a_i = 1. \tag{3}$$

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For the choice $s = 1,\ a_1 = 1$ and $\tau_1 = 1$, Eq. (1) becomes the Dickman equation

$$\dot{x}(t) = -\frac{1}{t}x(t-1), \tag{4}$$

well-known in number theory. The initial function $x(t) = 1,\ t \in [0,\ 1]$ defines a solution of (4), called the Dickman function (or Dickman-de Bruijn function) estimating the proportion of smooth numbers up to a given bound. We refer to such sources as [5,6,10,12,18,19] and to the references therein related to this equation. This is one of the reasons why we call (1) a generalized Dickman equation. Another reason is that the behavior of solutions of (1) preserves the typical structure properties as in the case of a Dickman equation.

The paper performs an asymptotic analysis of (1) in terms of the theory of dominant and subdominant positive solutions as it is developed in [11]. Let us recall the main statements of this paper necessary for our analysis. Consider the equation

$$\dot{x}(t) = -\sum_{i=1}^{s} c_i(t)x(t-\tau_i(t)), \tag{5}$$

which is more general than (4), where c_i: $[t_0,\ \infty) \to [0,\ \infty)$ and τ_i: $[t_0,\ \infty) \to (0,\ r]$ are continuous functions, and $\sum_{i=1}^{s} c_i(t) > 0,\ t \in [t_0,\ \infty)$.

Let us define some necessary concepts related to Eq. (5) (and obviously to its partial case – Eq. (1) as well). A continuous function $x : [t_0 - r, \infty) \to \mathbb{R}$ is called a solution of (5) on $[t_0 - r, \infty)$ if it is continuously differentiable on $[t_0,\ \infty)$ and satisfies (5) for every $t \in [t_0,\ \infty)$ (at $t = t_0$, the derivative is regarded as derivative on the right). The initial problem $x = \varphi(t)$, $t \in [t_0 - r, t_0)$, where φ is a continuous function, defines a unique solution $x = x(t_0, \varphi)(t)$, $t \geq t_0 - r$ of (5) such that $x(t_0, \varphi)(t) \equiv \varphi(t)$ if $t \in [t_0 - r, t_0]$.

A solution x of (5) on $[t_0 - r, \infty)$ is called positive if $x(t) > 0$ for every $t \in [t_0 - r, \infty)$, negative if $x(t) < 0$ for every $t \in [t_0 - r, \infty)$, and oscillating if it has arbitrarily large zeros on $[t_0 - r, \infty)$. In the case of $t \to \infty$, the symbols "o" and "O" used in formulas throughout the paper stand for the well-known Landau order symbols "small o" and "big O" and the symbol $\sim$ stands for the asymptotic equivalence. The same symbol is used when a function is decomposed into an asymptotic series.

The following two theorems are minor modifications of Theorem 8 and Theorem 9 of [11].

Theorem 1. *Let a positive solution of (5) exists on $[t_0,\ \infty)$. Then, there exist two positive solutions x_1 and x_2 (5) on $[t_0,\ \infty)$ satisfying*

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = 0 \tag{6}$$

such that every solution $x = x(t)$ of (5) on $[t_0,\ \infty)$ can be represented by the formula

$$x(t) = Kx_1(t) + O(x_2(t)), \tag{7}$$

where the constant K depends on x.

Theorem 2. *Let $\tilde{x}_1$ and $\tilde{x}_2$ be an arbitrary fixed pair of positive solutions of (5) on $[t_0,\ \infty)$ such that*

$$\lim_{t\to\infty} \frac{\tilde{x}_2(t)}{\tilde{x}_1(t)} = 0.$$

Then, the formula (7) remains valid if x_1 is replaced by $\tilde{x}_1$ and x_2 by $\tilde{x}_2$.

The following corollary is a simple consequence of Theorems 1 and 2.

Corollary 1. *There are no three eventually positive solutions x_1, x_2 and x_3 of (5) on $[t_0,\ \infty)$ such that*

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = 0, \quad \lim_{t\to\infty} \frac{x_3(t)}{x_2(t)} = 0.$$

Since, as it follows from the above two theorems, the existence of a positive solution of (5) on $[t_0,\ \infty)$ implies the existence of two positive solutions x_1 and x_2 of (5) on $[t_0,\ \infty)$ satisfying (6), and in (7), positive solutions x_1 and x_2 can be replaced by any other pair of solutions $\tilde{x}_1$ and $\tilde{x}_2$ of (5) positive on $[t_0,\ \infty)$, the solution x_1 is called a dominant solution and the solution x_2 is called a subdominant solution [11, Definition 2].

Although the structure formula (7) always holds (if a positive solution exists) and is, in a sense, a typical property of the solutions of Eq. (5), the problem of finding solutions x_1 and x_2 is usually unsolvable because it is often impossible to give an analytical formula for them. It is not easy even to describe at least the asymptotic behavior of x_1 and x_2 using asymptotic formulas or certain type of inequalities. We refer, for example, to the paper [16] on the asymptotic behavior of dominant solutions of the equation

$$\dot{x}(t) = p(t)x(t-\tau),$$

where $\tau \ \ 0$ and $p : [t_0, \infty) \to \mathbb{R}$ is a continuous function, and to [8,10,19] as well.