Applied Mathematics and Computation 332 (2018) 304–315

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Stochastic stabilization problem of complex networks without strong connectedness

Yan Liu, Jingling Mei, Wenxue Li*

Department of Mathematics, Harbin Institute of Technology, Weihai 264209, PR China

ARTICLE INFO

Keywords:
Stochastic stabilization
Connectedness
Complex networks
Coupled oscillators

ABSTRACT

This paper aims to study the stochastic stabilization problem of complex networks without strong connectedness (CNSC). To deal with large-scale complex networks which are not strongly connected, a hierarchical method and a hierarchical algorithm are given, respectively. Meanwhile, we construct a logarithmic Lyapunov function for CNSC. Next, based on the theory of asymptotically autonomous systems, the Lyapunov method and the graph theory, the whole complex network can be stabilized by stabilizing a part of nodes. Then, stability criteria of CNSC are given, whose conditions reflect the relationship between dynamic properties and topology structure clearly. Furthermore, the theoretical results are applied to coupled oscillators on CNSC to ensure the stability. Finally, a numerical example is provided to illustrate the effectiveness and practicability of the results.

1. Introduction

During the past several decades, complex networks which can be seen as digraphs [1–5] have become active topics due to their extensive applications in dealing with various issues in many fields, such as the World-Wide Web, neural networks, the food webs [6–9]. Meanwhile, many practical applications greatly depend on their dynamic properties, including the stability, the periodic oscillatory, the synchronization and so on. To the authors' knowledge, their dynamic properties can be affected by many factors, and the topology structure is one of the most important factors. Thus, the topology structure of complex networks has received a great deal of attention from researchers in investigating dynamic properties of large-scale natural and artificial networks, see [10–17].

On the other hand, as far as we know, various types of environmental noises exist in nature and they are commonly regarded as stochastic disturbances, which are usually considered as the sources of the instability, oscillations or poor performance. However, in fact, stochastic disturbances can not only be used to destabilize a stable network but also can stabilize an unstable network or make a stable network even more stable. For example, the linear scalar differential equation $\dot{x}(t) = \alpha x(t)$ $(\alpha > 0)$ is unstable, while it can be stabilized by a Brownian motion [18]. Thus, in order to research the stabilization problem of complex networks, it is an effective approach to take stochastic disturbances into account. Meanwhile, a great number of results on stochastic stabilization have been reported in literature [19–24]. Meanwhile, it should be pointed out that different interconnected nodes on complex networks may play different roles in stabilization issues and it is vital to categorize these nodes. However, most of existing papers only make effort to consider the effect of stochastic disturbances on the whole network. For instance, in [25], Kim et al. presented a robust H_∞ control to study the stability of time-delayed

* Corresponding author.
E-mail address: wenxuetg@hitwh.edu.cn (W. Li).

https://doi.org/10.1016/j.amc.2018.03.082
0096-3003/

systems with parameter uncertainties and stochastic disturbances; in [26], Zhao and Deng applied a new type of stability theorem for stochastic systems to consider stochastic stabilization. Therefore, studying whether the whole network can be stabilized by adding stochastic disturbances to a part of nodes of a network becomes significant.

In addition, we have witnessed an increasing interest for the stability of complex networks for several decades, since stochastic modeling has come to play an important role in many real networks. Meanwhile, many methods provide an effective tool to deal with relevant issues, such as the Lyapunov method, the robust control approach, the reciprocal method, etc. [3,27–29]. Nonetheless, progress is hindered by constructing an appropriate Lyapunov function directly. Luckily, in [3], Li and Shuai presented a systemic method that combined the Lyapunov method with the graph theory to construct a Lyapunov function. So far, lots of efforts have been devoted to applying this method to study the stability problems of complex networks [30–33]. We notice that the method in the literature mentioned above is based on the assumption that the network must be strongly connected. However, in fact, CNSC are more common in the real world. For example, the public by social interaction forms the social network, and neurons interconnected make up the neural network. The substation, transmission and distribution lines consist of the electric network, and goods by exchanging information online constitute the Internet of things. Thus, it is significant and meaningful to investigate the stabilization issues of CNSC.

Motivated by aforementioned discussions, in this paper, our purpose is to investigate the stochastic stabilization of CNSC. First, we can obtain a condensed network by taking each strongly connected component of the network as a single vertex and a hierarchical method for the condensed network is given. Then we provide a hierarchical algorithm accordingly. Moreover, stochastic disturbances are added to the first layer and a logarithmic Lyapunov function is constructed to investigate that the first layer achieves stability. Afterwards, based on the theory of asymptotically autonomous systems, the graph theory and the Lyapunov method, other layers can be stabilized with the effect of the first layer. Furthermore, we apply theoretical results to coupled oscillators and provide sufficient conditions for the stability. Finally, a numerical example is presented to illustrate the practicability of results obtained.

The main contributions of this paper can be summarized as follows. First, a hierarchical method and a hierarchical algorithm are given as effective tools to deal with CNSC, which can be realized by a mathematical software. Second, we construct a logarithmic Lyapunov function for CNSC and relax the assumption that the network should be strongly connected. Third, the stabilization problems of CNSC are studied by the theory of asymptotically autonomous systems, the Lyapunov method and the graph theory. The whole network can achieve stability by adding stochastic disturbances to a part of nodes of the network.

The rest of this paper is organized as follows. In Section 2, a hierarchical method and a hierarchical algorithm are outlined. In Section 3, our models which describe the network restricted on different strongly connected components are presented. Then we pay attention to deriving several sufficient conditions to ensure the stability of CNSC. In Section 4, an application is given about coupled oscillators to illustrate our results. Finally, a numerical example is provided in Section 5. A theorem and the corresponding proof are presented in the Appendix.

Notations. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and $B(t)$ is a one-dimensional Brownian motion defined on the space. Let $\mathbb{R}^n$ stand for n-dimensional Euclidean space. The superscript "T" stands for the transpose of a vector or a matrix. The notations $\mathbb{L} = \{1, 2, ..., l\}, \mathbb{Z}^+ = \{1, 2, ...\}, \mathbb{R}^1_+ = [0, +\infty)$ and $m = \sum_{i=1}^{l} m_i$ for $m_i \in \mathbb{Z}^+$ are used. Let $|\cdot|$ be the Euclidean norm for vectors or the trace norm of matrices. Define $||x||_{L^\infty} = \sup_{t\geq 0} |x(t)|$.

2. Hierarchical method for CNSC

In this section, in order to investigate CNSC, a hierarchical method and a hierarchical algorithm are given.

2.1. Hierarchical method

In this section, some fundamental concepts on the graph theory can be found in [34]. With regard to a digraph $\mathcal{G}$, a partial order $\preceq$ among vertices is to be defined at first. Let $V(\mathcal{G})$ denote the vertex set of a digraph. Without generality, let $V(\mathcal{G}) = \mathbb{L}$. For different vertices $k, h \in V(\mathcal{G}), k \preceq h$, if there is a directed path from vertex k to vertex h. Furthermore, we define a strongly connected component H of a digraph $\mathcal{G}$ as follows: if the subgraph H is strongly connected and for any vertex $k \notin V(H)$, the subgraph that consists of the vertex set $V(H) \cup \{k\}$ is not strongly connected, then H is a strongly connected component. Then we compress each strongly connected component of a digraph $\mathcal{G}$ into a single vertex, all of which constitute a new digraph $\mathcal{H}$ treated as a condensed digraph of $\mathcal{G}$ (see Fig. 1). Each vertex $H \in V(\mathcal{H})$ is a strongly connected component of a digraph $\mathcal{G}$.

Based on the above discussions, for a condensed digraph $\mathcal{H}$, we define a strict partial order $\prec$. For strongly connected components $H, \bar{H} \in V(\mathcal{H})$, if $H, \bar{H} \in V(\mathcal{H})$ satisfy both $H \prec \bar{H}$ and $\bar{H} \prec H$, then H and $\bar{H}$ are the same strongly connected components. Thus, relation $\prec$ is a strict partial order. We further obtain that for a condensed digraph $\mathcal{H}$ with a finite number of vertices, there must be both minimal and maximal elements in $V(\mathcal{H})$ with respect to the strict partial order $\prec$. As for the condensed digraph $\mathcal{H}$, we propose a hierarchical method throughout this paper as follows.

In the condensed digraph $\mathcal{H}$, all the vertices with zero in-degree belong to the first layer. Vertex M is called the vertex in kth $(k \geq 2)$ layer, if and only if the following conditions hold:

1. There at least exists a vertex N belonging to the $(k-1)$th layer satisfying that $N \prec M$.