Applied Mathematics and Computation 328 (2018) 276–286

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On causal extrapolation of sequences with applications to forecasting

Nikolai Dokuchaev[a,b,*]

[a] Department of Mathematics and Statistics, Curtin University, GPO Box U1987, Perth 6845, Western Australia, Australia
[b] Department of Information and Navigation Systems, National Research University ITMO, St. Petersburg 197101, Russia

ARTICLE INFO

MSC:
42A38
93E10
42A99

Keywords:
Smooth sequences
Extrapolation
Frequency analysis
Forecasting

ABSTRACT

The paper suggests a method of extrapolation of notion of one-sided semi-infinite sequences representing traces of two-sided band-limited sequences; this features ensure uniqueness of this extrapolation and possibility to use this for forecasting. This lead to a forecasting method for more general sequences without this feature based on minimization of the mean square error between the observed path and a predicable sequence. These procedure involves calculation of this predictable path; the procedure can be interpreted as causal smoothing. The corresponding smoothed sequences allow unique extrapolations to future times that can be interpreted as optimal forecasts.

1. Introduction

We study causal dynamic approximation and extrapolation of real sequences in deterministic setting, i.e. without probabilistic assumptions. Extrapolation of sequences can be used for forecasting and was studied intensively, for example, in the framework of system identification methods; see e.g. [23]. In signal processing, there is an approach oriented on the frequency analysis and exploring special features of the band-limited processes such as predictability. For stochastic stationary discrete time processes, the connection between predicability and degeneracy of the spectrum was established by the classical Szegö–Kolmogorov theorem; see a recent reviews in [4]. This theorem says that the optimal prediction error is zero if its spectral density is vanishing with a certain rate at a point of the unit circle $\mathbb{T}=\{z\in\mathbf{C}:\ |z|=1\}$, in particular, if it is vanishing on an arc on $\mathbb{T}$. In this case, the process is called “band-limited”. This result was expanded on more general stochastic processes featuring spectral densities; see, e.g., [6]. In deterministic pathwise setting without probability assumptions, this result was expanded [9,10] on sequences with Z-transform vanishing at a point of $\mathbb{T}$.

The present suggests to use the predicability featuring by band-limited processes for forecasting of more general processes that are not necessarily band-limited. This requires calculation of a trace of a band-limited process representing optimal approximation of the available observations. The extrapolation of this trace of a band-limited process can be used as a forecast. The motivation for that approach is based on the assumption that a band-limited part of a process can be interpreted as its regular part purified from a noise represented by high-frequency component. This leads to a problem of causal band-limited approximations for non-bandlimited processes which can be interpreted as a causal band-limited smoothing.

A known two-sided sequence can be converted into a band-limited process with a low-pass filter, and the resulting process will be an optimal band-limited approximation. However, an ideal low-pass filter is non-causal; therefore, it cannot

* Correspondence to: Department of Mathematics and Statistics, Curtin University, GPO Box U1987, Perth 6845, Western Australia, Australia
E-mail address: N.Dokuchaev@curtin.edu.au

https://doi.org/10.1016/j.amc.2018.01.038
0096-3003/

be applied for dynamically observable processes with unavailable future values which excludes predicting and extrapolation problems. Respectively, causal smoothing cannot convert a process into a band-limited one; it is known that the distance of an ideal low-pass filter from the set of all causal filters is positive [3]. There are many works devoted to causal smoothing and sampling, oriented on estimation and minimization of errors in L_2-norms or similar norms, especially in stochastic setting; see e.g. [1–3,7,8,11–13,16,19,20,25,27,28].

The present paper readdresses the problem of causal band-limited smoothing approximation and considers the problem of causal band-limited extrapolation for one-sided real sequences that are not necessary paths of band-limited processes.

We consider purely discrete time processes rather than samples of continuous time processes; one may say that the values between fixed discrete times are are not included into consideration. This setting imposes certain restrictions. In particular, it does not allow to consider continuously variable locations of the sampling points, as is common in sampling analysis of continuous time processes; see e.g. [5,16,18,21]. For continuous time processes, the predicting horizon can be selected to be arbitrarily small, such as in the model considered in [5]; this possibility is absent for discrete time processes considered below. In addition, it is not obvious how to define for discrete time processes or sequences an analog of the continuous time analyticity that is often associated with predicability.

Further, we consider the problem in the deterministic setting, i.e. pathwise. This means that the method has to rely on the intrinsic properties of a sole underlying sequence without appealing to statistical properties of an ensemble of sequences. In particular, we use a pathwise optimality criterion rather than criterions calculated via the expectation on a probability space such as mean variance criterions.

In addition, we consider an approximation that does not target the match of the values at any set of selected points; the error is not expected to be small. This is different from a more common setting where the goal is to match an approximating curve with the underlying process at certain sampling points; see e.g. [8,18,19,21]. Our setting is closer to the setting from [16,17,24,27,28]. In [16,17], the point-wise matching error was estimated for a sampling series and for a band-limited process representing smoothed underlying continuous time process; the estimate featured a given vanishing error. In [24], the problem of minimization of the total energy of the approximating bandlimited process was considered; this causal approximation was constructed within a given distance from the original process smoothed by an ideal low-pass filter. Another related result was obtained in [15], where an interpolation problem for absent sampling points was considered in a setting with vanishing error, for a finite number of sampling points. In [27], extrapolation of a trace of a band-limited process from a finite number of points was considered in a frequency setting for a general linear transform and some special Slepian's type basis in the frequency domain. In [28], a setting similar to [27] was considered for extrapolation of a trace of continuous time process from a finite interval using a special basis from eigenfunctions in the frequency domain. Our setting is different: we consider extrapolation without exact match of values for the underlying process. Therefore, we suggest to calculate extrapolations that can be used for forecasting that are not necessarily matching the values of the underlying process. This allows to consider semi-infinite underlying processes that are not paths of band-limited processes.

It can be noted that the framework of two-sided sequences required for detecting of the bandlimitness via Z-transforms are not always convenient to use. For example, consider a situation where the data is collected dynamically during a prolonged time interval. For many models, it is more convenient to represent this data flow as one-sided sequences such that $x(t)$ represents outdated observations with diminishing significance as $t\to-\infty$. However, application of the two-sided Z-transform requires to select some past time at the middle of the time interval of the observations as the zero point for a model of the two-sided sequence; this could be inconvenient. On the other hand, a straightforward application of the one-sided Z-transform to the historical data represented as one-sided sequences generates Z-transforms that cannot vanish on a part of the unit circle, even for traces of band-limited two-sided sequences. So far, the notion of bandlimitness was not expanded on the one-sided sequences $\{x(t)\}_{t=0,-1,-2,\ldots,-\infty}$. The paper addresses this problem, as well as the problem of detecting one-sided semi-infinite sequences that can be extended into two-sided band-limited processes (i.e. representing traces of band-limited processes).

The paper suggests a method of extrapolation of notion of one-sided semi-infinite sequences representing traces of two-sided band-limited sequences; this features ensure uniqueness of this extrapolation and possibility to use this for forecasting. This lead to a forecasting method for more general sequences without this feature based on minimization of the mean square error between the observed path and a predicable sequence. These procedure involves calculation of this predictable path; the procedure can be interpreted as causal smoothing. The corresponding smoothed sequences allow unique extrapolations to future times that can be interpreted as optimal forecasts.

For the solution, we use non-singularity of special sinc matrices obtained in [21] for the solution of the so-called superoscillations problem for continuous time processes; see the references in [18,21]. It can be noted that the setting in [18,21] considers exact matching of the band-limited process and the underlying process in certain points, which is different from our setting.

The sustainability of the method is demonstrated with some numerical experiments where we compare the band-limited extrapolation with some classical spline based interpolations.

2. Definitions

We use notation $\mathrm{sinc}\,(x)=\sin(x)/x$, and we denote by $\mathbb{Z}$ the set of all integers.