Applied Mathematics and Computation 327 (2018) 93–103

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Faint and clustered components in exponential analysis

Annie Cuyt[a], Min-nan Tsai[a,b], Marleen Verhoye[c], Wen-shin Lee[a,*]

[a] Department of Mathematics and Computer Science, University of Antwerp, Belgium
[b] Department of Computer Science and Information Engineering, National Taiwan Normal University, Taiwan
[c] Bio-Imaging Lab, University of Antwerp, Belgium

ARTICLE INFO

Keywords:
Multi-exponential analysis
Padé approximation
Spectral analysis

ABSTRACT

An important hurdle in multi-exponential analysis is the correct detection of the number of components in a multi-exponential signal and their subsequent identification. This is especially difficult if one or more of these terms are faint and/or covered by noise. We present an approach to tackle this problem and illustrate its usefulness in motor current signature analysis (MCSA), relaxometry (in FLIM and MRI) and magnetic resonance spectroscopy (MRS).

The approach is based on viewing the exponential analysis as a Padé approximation problem and makes use of some well-known theorems from Padé approximation theory. We show how to achieve a clear separation of signal and noise by computing sufficiently high order Padé approximants, thus modeling both the signal and the noise, rather than filtering out the noise at an earlier stage and return a low order approximant.

We illustrate the usefulness of the approach in different practical situations, where some exponential components are difficult to detect and retrieve because they are either faint compared to the other signal elements or contained in a cluster of similar exponential components.

1. Introduction

Many real-time experiments involve the measurement of signals which fall exponentially with time. The problem is then to determine, from such measurements, the number of components n and the value of all the parameters in the exponential model

$$\begin{aligned} \phi(t) &= \sum_{i=1}^{n} \alpha_i \exp(\phi_i t), \\ \alpha_i &= \beta_i \exp(\mathrm{i}\gamma_i), \quad \phi_i = \psi_i + \mathrm{i}\omega_i, \quad \mathrm{i}^2 = -1. \end{aligned} \tag{1}$$

Here ψ_i, ω_i, β_i and γ_i are respectively called the damping, frequency, amplitude and phase of each exponential term. The technique of multi-exponential analysis is closely related to what is commonly known in the applied sciences as the Padé–Laplace method [2] and the technique of sparse interpolation in the field of symbolic computation [14].

The basic method to estimate the parameters in a sum of complex exponentials is due to Prony [29]. It was later refined by Pisarenko [27]. Modern computer implementations include the MUltiple SIgnal Classification algorithm MUSIC [31], the

* Corresponding author.
E-mail addresses: annie.cuyt@uantwerpen.be (A. Cuyt), edwen0224@gmail.com (M.-n. Tsai), marleen.verhoye@uantwerpen.be (M. Verhoye), wen-shin.lee@uantwerpen.be (W.-s. Lee).

https://doi.org/10.1016/j.amc.2017.11.007
0096-3003/

Estimation of Signal Parameters via Rotational Invariance Techniques ESPRIT [30], the Toeplitz Approximation Method TAM [23] and the Matrix Pencil method MP [19].

The major hurdles in multi-exponential modelling are the correct detection of the number of components in the model, the noise sensitivity of problem statement and computational method, the limitation to distinguish closely spaced frequencies ω_i [8], and the difficulty in resolving different exponential decays ψ_i if the damping factors are too much alike [4].

Before we proceed, some words on Fourier methods are appropriate. Fourier analysis is not very well suited for the decomposition of aperiodic signals, such as exponentially decaying ones. The damping causes a broadening of the spectral peaks, which in its turn leads to the peaks overlapping and masking the smaller amplitude peaks. The latter are important for the fine level signal classification. Moreover, Fourier analysis completely ignores any actual physical multi-exponential structure of the encoded time signal.

Here we show how to make use of the connections between signal processing, Padé approximation and sparse interpolation to:

- retrieve the correct number of components in a signal.
- and the characteristics of faint components buried in noise.

How to regularize the exponential analysis problem statement and improve the ω_i- and ψ_i-resolution is dealt with in [10–12].

2. Exponential analysis and sparse interpolation

Let us assume that in $\phi(t)$ given by (1), the frequency content is limited by the bandwidth Ω,

$$|\Im(\phi_i)| = |\omega_i| < \Omega/2, \qquad i = 1, \dots, n,$$

where $\Im(\cdot)$ denotes the imaginary part of a complex number. Also, let $\phi(t)$ be sampled at the equidistant points $t_j = j\Delta$ for $j = 0, 1, \dots, 2n-1, \dots,$ with $\Delta < 2\pi/\Omega$ [25,32] in order to avoid aliasing effects in the analysis and reconstruction of $\phi(t)$. In the sequel we denote

$$f_j := \phi(t_j), \qquad j = 0, 1, \dots, 2n-1, \dots$$

We now summarize the exponential analysis method that underlies all modern implementations to extract $n, \phi_1, \dots, \phi_n, \alpha_1, \dots, \alpha_n$ from the uniformly taken measurements $f_0, \dots, f_{2n-1}, \dots$ and the specific form (1) of the model for $\phi(t)$.

2.1. Foundations of Prony's method

If we further denote

$$\lambda_i := \exp(\phi_i \Delta) = (\exp(\Delta))^{\phi_i},$$

then it is apparent that the data f_j are structured, namely

$$f_j = \sum_{i=1}^{n} \alpha_i \lambda_i^j, \qquad j = 0, \dots, 2n-1, \dots \tag{2}$$

The system of equations (2) is called a sparse interpolation problem: the data f_j taken at the equidistant points $t_j = j\Delta$ are interpolated by the expression

$$\sum_{i=1}^{n} \alpha_i (\exp(t))^{\phi_i},$$

where the $\alpha_i, \phi_i, i = 1, \dots, n$ are unknown.

Let us define the Hankel matrices

$$H_n^{(r)} := \begin{pmatrix} f_r & \cdots & f_{r+n-1} \\ \vdots & \iddots & \vdots \\ f_{r+n-1} & \cdots & f_{r+2n-2} \end{pmatrix}, \qquad r \geq 0.$$

It is known from [18, p. 603] that for the multi-exponential samples f_j given by (2),

$$\det H_\nu^{(r)} = 0, \quad \nu > n, \quad r \geq 0 \tag{3}$$

$$\det H_n^{(r)} \neq 0, \quad r \geq 0 \tag{4}$$

and it is proved in [21] that

$$\det H_\nu^{(r)} = 0 \text{ only accidentally}, \quad \nu < n. \tag{5}$$