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Two projection methods for the solution of Signorini problems

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A B S T R A C T

Two iterative methods, based on projection and boundary element methods, are considered for Signorini problems. The regularized problem with the projection boundary condition is first deduced from the Signorini problem. By using the equivalence between the Signorini boundary condition and the projection fixed point problem, our methods formulate the Signorini boundary condition into a sequence of Robin boundary conditions. In the new boundary condition there is a penalty parameter ρ. The convergence speed of the first method is greatly influenced by the value of ρ which is difficult to choose for individual problems. To improve the performance of this method, we present a self-adaptive projection method which adjusts the parameter ρ automatically per iteration based on the iterative data. The main result of this work is to provide the convergence of the methods under mild assumptions. As the iteration process is given by the potential and its derivative on the boundary of the domain, the unknowns of the problem are computed explicitly by using the boundary element method. Both theoretical results and numerical experiments indicate efficiency of the methods proposed.

1. Introduction

It is well known that elliptic Signorini problems are important and very useful class of nonlinear problems arising from physics, engineering etc. Such problems are difficult to solve because the classical Dirichlet, Neumann and Robin boundary conditions are unknown on the Signorini boundary in advance. The existence and uniqueness of the solution are established by formulating the problem as variational inequalities [5]. Since the Signorini condition is imposed at the boundary of the domain, the boundary methods, such as the method of fundamental solutions (MFS) and the boundary element method (BEM), are suitable for the solution of Signorini problems. There are several efficient algorithms in conjunction with these boundary methods for the numerical solution. For example, the Signorini problem was formulated as a constrained minimization problem by the MFS [2,4,13,23,24]. For the BEM, it seems more suitable for the solution because the unknowns (the potential and its normal derivative) on the Signorini boundary are related directly by a linear system [1,6,14,15,18,19,28,29].

It is worth mentioning that the iterative algorithm has become an efficient numerical technique for finding the approximate solution of Signorini problems [1,11,12,15,25–29,31]. Among these methods, semi-smooth Newton methods and augmented Lagrangian methods are extensively applied to variational inequalities and Signorini problems [11,12,25–27,29,31]. However, these methods have to solve a nonlinear problem in every iteration step, and the penalty parameter ρ of the methods must be sufficiently large. Using the BEM, we also developed several projection iterative methods [32–34] for the

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Signorini problem, which formulate the Signorini boundary condition into a sequence of simple boundary conditions such as Dirichlet, Neumann and Robin boundary conditions.

In the last two decades, a class of iterative methods, named implicit method, emerged for the finite dimensional variational inequalities. The basic idea is contained in [8,20–22], while the convergence depends significantly on the penalty parameter ρ. Moreover, the proper parameters are different for individual problems and difficult to choose. To overcome the disadvantage, some self-adaptive projection methods were used to adjust the parameter automatically [7,9,16,17]. Motivated by the above methods, we propose two projection methods for solving the Signorini problem in infinite-dimensional spaces. The methods convert the Signorini boundary condition into a series of Robin boundary conditions and are implemented in conjunction with the BEM [3,10,30], then our methods are different from other methods and only require to solve a linear variational problem for each iteration.

The structure of this paper is organized as follows. In Section 2, we establish the equivalence between the Signorini boundary condition and a fixed point problem using the projection method, and we introduce a weak formulation for the Signorini problem. Sections 3 and 4 are devoted to the convergence analysis of the two new methods: the projection method and self-adaptive projection method, respectively. Section 5 deals with the numerical implementation of combining the methods with the BEM for the solution. Furthermore, the self-adaptive adjustment rule is presented for the second method. In Section 6, we present some numerical results to illustrate and compare the performance of the methods. Finally, some concluding remarks are given in Section 7.

2. Formulation of the Signorini problem

Let $\Omega \subset \mathbb{R}^2$ an open and bounded domain with a smooth boundary $\Gamma = \partial\Omega$ which consists of three disjoint parts Γ_D, Γ_N and Γ_S, where Dirichlet, Neumann and Signorini boundary conditions are given respectively. In this paper we consider the Signorini problem: given $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma \setminus \bar{\Gamma}_N)$ and $h \in H^{-1/2}(\Gamma \setminus \bar{\Gamma}_D)$, find $u \in H^1(\Omega)$ such that

$$\Delta u = f \text{ in } \Omega, \tag{1}$$

$$u = g \text{ on } \Gamma_D, \tag{2}$$

$$\lambda = h \text{ on } \Gamma_N, \tag{3}$$

$$u \geq g,\ \ \lambda \geq h,\ \ (u-g)(\lambda-h) = 0 \text{ on } \Gamma_S. \tag{4}$$

where $\lambda := \frac{\partial u}{\partial n}$ and n denotes the outward unit normal to the boundary. To give the weak formulation of the above problem, we introduce the following space of functions

$$H_g^1(\Omega) := \left\{ v \in H^1(\Omega),\ \ v = g \text{ on } \Gamma_D \right\},$$

$$K := \left\{ v \in H_g^1(\Omega),\ \ v \geq g \text{ on } \Gamma_S \right\}.$$

It is well known that the Signorini problem is then equivalent to a variational inequality

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ \int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_{\Gamma_N \cup \Gamma_S} h(v-u)ds + \int_\Omega f(v-u)dx,\ \ \forall v \in K. \end{cases} \tag{5}$$

or a minimization problem

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ J(u) = \min_{v \in K} J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_{\Gamma_N \cup \Gamma_S} hv ds - \int_\Omega fv dx. \end{cases} \tag{6}$$

We assume that the problem satisfies $\Gamma_D \neq \varnothing$ or $\int_{\Gamma_N \cup \Gamma_S} hds < 0$. Then the problem (5) or equivalently (6) admits a unique solution in K that depends continuously on the data f, g and h [5,6,12,26–28,35].

Remark 2.1. A large number of fluid mechanics and heat transfer problems can be modeled by Signorini problems, as a simplified version of a problem occurring in elasticity, the simplified Signorini problem [11,25,26] consists of finding $u \in H^1(\Omega)$ satisfying

$$\begin{cases} -\Delta u + u = f \text{ in } \Omega, \\ u \geq 0,\ \lambda \geq h,\ u(\lambda - h) = 0 \text{ on } \Gamma. \end{cases}$$