Applied Mathematics and Computation 326 (2018) 117–123

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the vertex partitions of sparse graphs into an independent vertex set and a forest with bounded maximum degree☆

Min Chen*, Weiqiang Yu, Weifan Wang

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

ARTICLE INFO

Keywords:
Vertex partition
Maximum average degree
Forest
Girth

ABSTRACT

Given a graph $G = (V, E)$, if its vertex set $V(G)$ can be partitioned into two non-empty subsets V_1 and V_2 such that $G[V_1]$ is edgeless and $G[V_2]$ is a graph with maximum degree at most k, then we say that G admits an (I, Δ_k)-partition. A similar definition can be given for the notation (I, F_k)-partition if $G[V_2]$ is a forest with maximum degree at most k.

The maximum average degree of G is defined to be $\mathrm{mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$. Borodin and Kostochka (2014) proved that every graph G with $\mathrm{mad}(G) \le \frac{8}{3}$ admits an (I, Δ_2)-partition and every graph G with $\mathrm{mad}(G) \le \frac{14}{5}$ admits an (I, Δ_4)-partition. In this paper, we obtain a strengthening result by showing that for any $k \ge 2$, every graph G with $\mathrm{mad}(G) \le 2 + \frac{k}{k+1}$ admits an (I, F_k)-partition. As a corollary, every planar graph with girth at least 7 admits an (I, F_4)-partition and every planar graph with girth at least 8 admits an (I, F_2)-partition. The later result is best possible since neither girth condition nor the class of F_2 can be further improved.

1. Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. Let $G = (V, E)$ be a graph. Denote m classes by graphs $G_1, \ldots, G_m$. Call a vertex partition of G a $(G_1, \ldots, G_m)$*-partition* if $V(G)$ can be partitioned into m sets $V_1, \ldots, V_m$ such that for each $1 \le l \le m$, the subgraph $G[V_l]$ belongs to G_l. For simplicity, we denote the class of forests, the class of independent sets, the class of graphs with maximum degree k, and the class of forests with maximum degree k by using notation F, I, Δ_k and F_k, respectively. Let $g(G)$ denote the *girth* of G which is the length of a shortest cycle in G.

In recent years, several papers concerning vertex partitions of graphs with some restrictions on sparseness or girth condition have appeared. The *maximum average degree* of G is defined to be $\mathrm{mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$. It is used to measure the sparseness of G. Kurek and Ruciński [10] defined that graphs with low maximum average degree are said to be *globally sparse*.

The following lemma is folklore and it connects the relationship between the girth of a planar graph and maximum average degree.

Lemma 1. *If G is a connected planar graph, then* $mad(G) < \frac{2g(G)}{g(G)-2}$.

☆ Research supported by NSFC (Nos. 11471293, 11771402).

* Corresponding author.
E-mail addresses: chenmin@zjnu.cn (M. Chen), wwf@zjnu.cn (W. Wang).

https://doi.org/10.1016/j.amc.2018.01.003
0096-3003/

The famous four-color theorem [1,2] guarantees that every planar graph has an (I, I, I, I)-partition. By investigating acyclic coloring problems, Borodin [3] showed that every planar graph has an (I, F, F)-partition. At the same time, in [4], Borodin and Glebov showed that every planar graph with girth $g \geq 5$ has an (I, F)-partition. Moreover, Borodin et al. [5] constructed a planar graph with girth $g = 6$ having no (I, Δ_k)-partition for any k. Naturally, it seems to be interesting to study (I, Δ_k)-partition (or (I, F_k)-partition) problems for planar graphs with girth $g \geq 7$. Notice that $\Delta_1 = F_1$.

Borodin and Kostochka [6], in 2011, proved that every graph G satisfying $\mathrm{mad}(G) < \frac{12}{5}$ admits an (I, F_1)-partition. By applying Lemma 1, this implies that every planar graph with girth $g \geq 12$ admits an (I, F_1)-partition. This has been later improved by Kim et at. [9] in which they showed that every planar graph with girth $g \geq 11$ admits an (I, F_1)-partition. Since there exist non-(I, F_1)-partitioning planar graphs with girth $g = 9$, it is worthy of thinking the (I, F_1)-partition problems of planar graphs with girth $g = 10$.

On the other hand, Borodin and Kostochka [7] obtained that every graph G satisfying $\mathrm{mad}(G) \leq \frac{8}{3}$ admits an (I, Δ_2)-partition and every graph G satisfying $\mathrm{mad}(G) \leq \frac{14}{5}$ admits an (I, Δ_4)-partition. Again, by Lemma 1, this yields that every planar graph with girth $g \geq 7$ admits an (I, Δ_4)-partition and every planar graph with girth $g \geq 8$ admits an (I, Δ_2)-partition. Montassier and Ochem [11] proved that deciding if a planar graph of girth $g \geq 7$ has an (I, Δ_2)-partition is NP-complete. Recently, Dross et al. [8] considered (I, F_k)-partition problems of the same family of planar graphs. They showed that every planar graph with girth $g \geq 7$ admits an (I, F_5)-partition and every planar graph with girth $g \geq 8$ admits an (I, F_3)-partition.

In this paper, we prove the following:

Theorem 1. *For $k \geq 2$, every graph G with $mad(G) \leq 2 + \frac{k}{k+1}$ admits an (I, F_k)-partition.*

Theorem 1 together with Lemma 1 yields Corollary 1.

Corollary 1. *Each planar graph G admits:*

(1) *an (I, F_4)-partition if $g(G) \geq 7$;*
(2) *an (I, F_2)-partition if $g(G) \geq 8$.*

Corollary 1 is a common strengthening of some previous results in [7,8]. Moreover, the second result of Corollary 1 is best possible since neither girth condition nor the class of F_2 can be improved. Still, we suspect that Corollary 1 (1) can be further strengthened.

In fact, to derive Theorem 1, we will need a more precise statement. For a given graph G and vertex subset $S \subseteq V(G)$, we define

$$\rho(S, G) := \left(1 + \frac{k}{2k+2}\right)|S| - |E(G[S])|. \tag{1}$$

By definition, we see that $\mathrm{mad}(G) \leq 2 + \frac{k}{k+1}$ is equivalent to the assumption that $\rho(S, G) \geq 0$ for each $S \subseteq V(G)$. Our main result is the following which is stronger than Theorem 1.

Theorem 2. *For $k \geq 2$, if a graph G satisfies that*

$$\rho(S, G) > -\frac{1}{k+1} \quad for \quad every \quad S \subseteq V(G), \tag{2}$$

then G admits an (I, F_k)-partition.

2. Proof of Theorem 2

2.1. Some basic notation

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, the *degree* of v is denoted by $d_G(v)$. Call v an *m-vertex*, or an *m^+-vertex*, or an *m^--vertex* if $d_G(v) = m$, or $d_G(v) \geq m$, or $d_G(v) \leq m$. Let $V_p(G)$ denote the set of p-vertices in G. A triangle is called *special* if it contains at least two 2-vertices. Meanwhile, these kinds of 2-vertices are both called *special*. By a *host* vertex we mean a 3^+-vertex $v \in V(G)$ which is incident to at least one special triangle. If a host vertex is incident to exactly $i+1$ special triangles, then it is said to be an *i-host* vertex. Obviously, an i-host vertex has degree at least $2i+2$. Furthermore, we denote by G^* the graph which is obtained from G by deleting all special 2-vertices and all degree 1^- vertices.

Definition 1. A graph H is *smaller than* a graph G if

(i) $|V_{2^+}(H^*)| < |V_{2^+}(G^*)|$, or if
(ii) $|V_{2^+}(H^*)| = |V_{2^+}(G^*)|$ and $|V(H^*)| < |V(G^*)|$, or if
(iii) $|V_{2^+}(H^*)| = |V_{2^+}(G^*)|$, $|V(H^*)| = |V(G^*)|$ and $|E(H^*)| < |E(G^*)|$, or if
(iv) $|V_{2^+}(H^*)| = |V_{2^+}(G^*)|$, $|V(H^*)| = |V(G^*)|$, $|E(H^*)| = |E(G^*)|$, and $|V(H)| < |V(G)|$.