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High-order full discretization for anisotropic wave equations

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ABSTRACT

Two-dimensional linear wave equation in anisotropic media, on a rectangular domain with initial conditions and periodic boundary conditions, is considered. The energy of the problem is contemplated. The space discretization is reached by means of finite differences on a uniform grid, paying attention to the mixed derivative of the equation. The discrete energy of the semi-discrete problem is introduced. For the time integration of the system of ordinary differential equations obtained, a fourth order exponential splitting method, which is a geometric integrator, is proposed. This time integrator is efficient and easy to implement. The stability condition for time step and space step ratio is deduced. Numerical experiments displaying the good behavior in the long time integration and the efficiency of the numerical solution are provided.

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1. Introduction

Anisotropic media, in which the velocity may depend on the direction, are important in several wave propagation models, as anisotropic Maxwell's equations [3] or in elastic anisotropic waves in solid-earth geophysics [11]. Anisotropy seems to be an everywhere property of earth materials and its effects on seismic data must be taken into account. Today, seismic anisotropy is considered in exploration and reservoir characterization [24]. Stability analysis of the Perfectly Matched Layer method applied to anisotropic waves in two dimensions are studied for example in [4,17,21].

In this paper we study a particular case of the equation considered in [5], the two dimensional time-dependent anisotropic and dispersive wave equation

$$\partial_{tt}u = \alpha_{11}\partial_{xx}u + 2\alpha_{12}\partial_{xy}u + \alpha_{22}\partial_{yy}u - s^2u. \tag{1}$$

We assume that the coefficients α_{ij} and s^2 in (1) are constant satisfying

$$\alpha_{11} > 0,\ \alpha_{22} > 0,\ \alpha_{11}\alpha_{22} - \alpha_{12}^2 > 0, \tag{2}$$

so that in the steady state the equation is elliptic.

When a problem posed in an infinite domain is solved numerically, it is necessary to reduce the computational domain to a finite domain, which forces us to choose suitable boundary conditions. On physical applications, it is desirable to have numerical models that resemble the dynamics of the continuous problems. If periodic boundary conditions are taken, invariants of the original problem are preserved. Here, we consider Eq. (1) in a rectangular domain $R = [a, b] \times [c, d]$, for the unknown $u(x, y, t)$, with periodic boundary conditions,

$$u(a, y, t) = u(b, y, t), \qquad y \in [c, d], \tag{3}$$

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$$\partial_x u(a, y, t) = \partial_x u(b, y, t), \quad y \in [c, d], \tag{4}$$

$$u(x, c, t) = u(x, d, t), \qquad x \in [a, b], \tag{5}$$

$$\partial_y u(x, c, t) = \partial_y u(x, d, t), \quad x \in [a, b]. \tag{6}$$

and initial conditions,

$$u(x, y, 0) = u_0(x, y), \quad \partial_t u(x, y, 0) = v_0(x, y), \tag{7}$$

which satisfy the periodic boundary conditions in R.

In an isotropic medium, $\alpha_{12} = 0$, $\alpha_{11} = \alpha_{22}$, we get the Klein–Gordon wave equation. If in the Eq. (1), $\alpha_{12} = 0$ but $\alpha_{11} \neq \alpha_{22}$, there are different speeds on x direction and on y direction, which corresponds to the orthotropic case. However, the general anisotropic case occurs when $\alpha_{12} \neq 0$. This means the existence of a spatial mixed derivative term in (1). In the literature there are other problems containing spatial mixed derivative terms as convection–diffusion equations [8,13,14], parabolic problems with application to pricing options [12,15,26] or in numerical mathematics when coordinate transformations are applied to allow working on simple domains or on uniform grids. In [13,14] the spatial derivatives are approximated by means of second-order finite differences, whereas in [8,12] fourth order finite differences are used. Then, the semi-discrete system of ordinary differential equations (ODEs) is integrated using alternating direction implicit schemes of first and second orders. For hyperbolic problems, as (1), is less common to use implicit methods because the stability condition is less demanding and $\triangle t$ and $\triangle x$ are of similar magnitude.

We are interested in obtaining efficient high order in space and time schemes for the numerical solution of Eq. (1), with periodic boundary conditions (3)–(6) and initial conditions (7). In this paper, the spatial derivatives are approximated using second and fourth order finite differences. As the boundary conditions are periodic, the matrix in the ODE system achieved is a block circulant matrix where each block is too a circulant matrix. For second order approximation of the spatial derivatives we prove that this matrix is symmetric negative definite and we locate the interval that contains its eigenvalues. We study well-posedness by using the discrete energy associated to the problem. For fourth order approximation of the spatial derivatives we compute numerically the eigenvalues of the corresponding symmetric matrix for moderate values of the dimension of the matrix, and the eigenvalues obtained are negative values.

We rewrite the semi-discrete problem as first order in time and the resulting ODE system is a Hamiltonian problem. This ODE system is split in two intermediate problems which are solved exactly. A fourth order splitting scheme is achieved by the flow composition of the two intermediate problems chosen. In stead of using alternating directions as in [8], the contribution of all spatial derivatives are regarded together because that the splitting obtained is computationally more efficient. A similar splitting method is considered in [2] for an isotropic problem with absorbing boundary conditions. The stability interval of the splitting method and the stability condition for the ratio between the time step and the space step are studied.

Useful overviews of splitting methods can be found in the review papers [6,19,20]. Splitting schemes are especially useful in the scope of geometric integration. Actually, splitting integrators preserve structural properties of the original problem's flow as long as the intermediate problems' flow do. The good performance of the geometric integrators in the long time integration of Hamiltonian ODE systems is well showed in [10,22].

The paper is organized as follows. The energy of the continuous problem is introduced in Section 2. In Section 3, second order approximation of the spatial derivatives are considered and the corresponding discrete energy is regarded. Section 4 is devoted to the exponential splitting time integrator. In Section 5 fourth order approximation of the spatial derivatives are introduced. Numerical experiments are conducted in Section 6. The good long time behavior as well as the efficiency of the splitting scheme by comparing with the fourth-order four-stage Runge–Kutta method in terms of CPU time are displayed.

2. Energy of the continuous problem

Knowing the energy of the system is important because it allows knowing an amount that is conserved over time without solving the equation. Moreover, when the continuous problem is discretized in space, we can compare the energy of the continuous problem with the energy of the semi-discrete problem.

An energy,

$$E(t) = \frac{1}{2} \iint_R ((\partial_t u(x, y, t))^2 + \alpha_{11} (\partial_x u(x, y, t))^2 + 2\alpha_{12} \partial_x u(x, y, t) \partial_y u(x, y, t) + \alpha_{22} (\partial_y u(x, y, t))^2 + s^2 u(x, y, t)^2) dxdy,$$

can be introduced. Here

$$(u, v) = \iint_R v^* u \, dxdy, \quad \|u\|^2 = (u, u).$$

Then,

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