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Characteristic conic of rational bilinear map

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ABSTRACT

Bilinear and rational bilinear maps defined by planar quadrilaterals play important roles in computer graphics and geometric design. We show that the inverse of a rational bilinear map has a close relationship with a conic, which is named as "characteristic conic" in this paper. We show that the characteristic conic of a rational bilinear map can be expressed as the rational quadratic Bézier curve, with its control points and weights in a closed form. Furthermore, we show that the characteristic conic is an envelope of two one-parameter families of straight lines, and all characteristic conics of a fixed quadrilateral form a one-parameter family.

1. Introduction

The bilinear map

$$\boldsymbol{v}(s,t) := (1-s)(1-t)\boldsymbol{v}_1 + s(1-t)\boldsymbol{v}_2 + st\boldsymbol{v}_3 + (1-s)t\boldsymbol{v}_4$$

plays an important role in computer graphics and geometric design [1], where $\boldsymbol{v}_i = (x_i, y_i)$, $i = 1, \ldots, 4$ are four vertices of a planar quadrilateral $\mathcal{Q}$ and s, t are the parameters. In general, the inverse of the map involves a square root [1,2] and thus the map is 2:1 over complex numbers. The question is whether these two pre-images are real and distinct, real and identical, or complex conjugates.

Here we give an example. For the bilinear map defined by $\mathcal{Q}$ with vertices $\boldsymbol{v}_1 = (0,0)$, $\boldsymbol{v}_2 = (12,0)$, $\boldsymbol{v}_3 = (12,4)$, $\boldsymbol{v}_4 = (8,6)$, the inverse of $\boldsymbol{v} = (x,y)$ is [3]

$$s = \frac{x - 4y + 36 \pm \sqrt{f(x,y)}}{24}, \quad t = \frac{-x + 4y + 36 \pm \sqrt{f(x,y)}}{32},$$

where

$$f(x,y) = x^2 - 8xy + 16y^2 - 72x - 96y + 1296. \tag{1}$$

Then $\boldsymbol{v} = (x,y)$ has 0, 1, or 2 real pre-images corresponding to three cases $f(x,y) < 0$, $f(x,y) = 0$ or $f(x,y) > 0$, respectively. (1) shows that the image of $f(x,y) = 0$ is a parabola, then $\boldsymbol{v} = (x,y)$ has 0, 1, or 2 real pre-images corresponding to the following three cases, $\boldsymbol{v}$ lies inside, on or outside of the parabola, respectively.

Recently, Sederberg and Zheng [3] studied the rational bilinear map

$$\boldsymbol{v}(s,t) = \frac{(1-s)(1-t)w_1\boldsymbol{v}_1 + s(1-t)w_2\boldsymbol{v}_2 + stw_3\boldsymbol{v}_3 + (1-s)tw_4\boldsymbol{v}_4}{(1-s)(1-t)w_1 + s(1-t)w_2 + stw_3 + (1-s)tw_4}. \tag{2}$$

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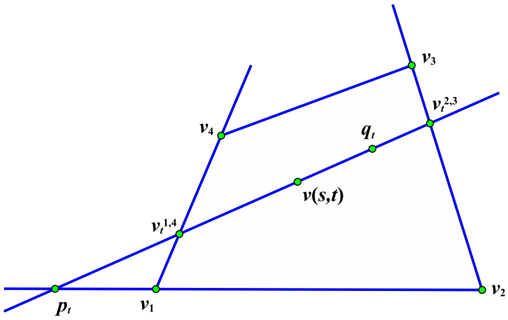

Fig. 1. Illustrations of quadrilateral and related vertices.

Let $C_i (i = 1, 2, 3, 4)$ be the triangle area,

$$C_i = A(\boldsymbol{v}_{i-1}, \boldsymbol{v}_i, \boldsymbol{v}_{i+1}), \tag{3}$$

where vertices are indexed cyclically, $\boldsymbol{v}_{i+4} = \boldsymbol{v}_i$, $i \in \mathbb{Z}$. They showed that if

$$\frac{w_1 w_3}{w_2 w_4} = \frac{C_1 C_3}{C_2 C_4}, \tag{4}$$

s and t are rational functions of $\boldsymbol{v}$, then the rational bilinear map is 1:1. In other words, $\boldsymbol{v}(s, t)$ has, and only has one real pre-image.

Let

$$A_i = A_i(\boldsymbol{v}) = A(\boldsymbol{v}, \boldsymbol{v}_i, \boldsymbol{v}_{i+1}),$$
$$B_i = B_i(\boldsymbol{v}) = A(\boldsymbol{v}, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}),$$
$$\tilde{A}_i = w_i w_{i+1} A_i, \tilde{B}_i = w_{i-1} w_{i+1} B_i,$$

Floater [2] derived the expression of s and t with arbitrary weights w_i as follows:

$$s = \frac{2\tilde{A}_4}{2\tilde{A}_4 - \tilde{B}_1 + \tilde{B}_2 \pm \sqrt{\tilde{D}}}, t = \frac{2\tilde{A}_1}{2\tilde{A}_1 - \tilde{B}_1 - \tilde{B}_2 \pm \sqrt{\tilde{D}}}, \tag{5}$$

where

$$\tilde{D} = \tilde{B}_1^2 + \tilde{B}_2^2 + 2\tilde{A}_1\tilde{A}_3 + 2\tilde{A}_2\tilde{A}_4. \tag{6}$$

Both $\tilde{A}_i$ and $\tilde{B}_i$ are linear functions of x, y, so $\tilde{D}$ is the quadratic formula of x, y and then the image of $\tilde{D} = 0$ is a conic. Thus, similar to the case of bilinear map, for rational bilinear map, $\boldsymbol{v} = (x, y)$ has 0, 1, or 2 real pre-images if $\boldsymbol{v}$ lies inside, on or outside of a conic, respectively.

The inverse of the bilinear map as well as the rational bilinear map shows that, relation between $\boldsymbol{v}$ and a conic determines the number of real pre-images of $\boldsymbol{v} = (x, y)$. Such a conic is called "characteristic conic" of a (rational) bilinear map because of its important role in the (rational) bilinear map.

Nevertheless, it is not intuitional to obtain geometric information of such conics from (6). In this paper we represent characteristic conics as rational quadratic Bézier curves and derive the closed form of their control points and weights, by which we get intuitional geometric information of characteristic conics. Furthermore, we show that the characteristic conic is an envelope of two one-parameter families of straight lines, while all characteristic conics of a fixed quadrilateral form a one-parameter family.

2. Inverse of rational bilinear map

In this section we discuss how to derive the inverse of a rational bilinear map. If the quadrilateral is a trapezoid, then each point has one pre-image [3]. So we only consider the case that the quadrilateral $\mathcal{Q}$ is not a trapezoid.

Obviously, if $\boldsymbol{v}(s, t)$ can be represented by (2), (s, t) is a pre-image of $\boldsymbol{v}(s, t)$. To find another possible pre-image of $\boldsymbol{v}(s, t)$, we select two points (see Fig. 1)

$$\boldsymbol{v}_t^{1,4} := \frac{(1-t)w_1\boldsymbol{v}_1 + tw_4\boldsymbol{v}_4}{(1-t)w_1 + tw_4}, \boldsymbol{v}_t^{2,3} := \frac{(1-t)w_2\boldsymbol{v}_2 + tw_3\boldsymbol{v}_3}{(1-t)w_2 + tw_3}, \tag{7}$$

from lines $\boldsymbol{v}_1\boldsymbol{v}_4$, $\boldsymbol{v}_2\boldsymbol{v}_3$, respectively.