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On the power method for quaternion right eigenvalue problem[☆]

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ABSTRACT

In this paper, we study the power method of the right eigenvalue problem of a quaternion matrix A. If A is Hermitian, we propose the power method that is a direct generalization of that of complex Hermitian matrix. When A is non-Hermitian, by applying the properties of quaternion right eigenvalues, we propose the power method for computing the standard right eigenvalue with the maximum norm and the associated eigenvector. We also briefly discuss the inverse power method and shift inverse power method for the both cases. The real structure-preserving algorithm of the power method in the two cases are also proposed, and numerical examples are provided to illustrate the efficiency of the proposed power method and inverse power method.

1. Introduction

Quaternion and quaternion matrices [1] have wide applications in applied science, such as field theory [2], Lagrangian formalism [3] and signal processing [4,5]. With the rapid development of the above disciplines, it is getting more and more necessary for us to further study the theoretical properties and numerical computations of quaternion and quaternion matrices.

In [6], Ding studied the transformations and relationships between some special matrices. In [7] and [8], Ding studied the methods of computing matrix exponentials and transition matrices related to linear dynamic systems and a Kalman filter based least squares iterative algorithm, respectively.

For real structure-preserving methods for quaternion matrices and applications, in [9–12], we proposed fast structure-preserving methods for computing the singular value decomposition, Householder based transformations, LU decomposition and Cholesky decomposition for quaternion matrices, respectively. And in [13], we proposed a new double color image watermarking algorithm based on the SVD and Arnold scrambling. In [14], we also compared two SVD-based color image compression schemes.

For real structure-preserving methods for quaternion right eigenvalue problem, in [15–17], the authors proposed real structure-preserving methods for computing right eigenvalues of quaternion Hermitian matrix, QR algorithm for quaternion non-Hermitian right eigenvalue problem and Jacobi algorithm for quaternion Hermitian right eigenvalue problem, respectively.

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The algebraic eigenvalue problem is one of the major problems in matrix computations, since it has many applications such as Pagerank, the important technique used in Google Search engines. For a real or complex matrices A, eigenvalue problem is well studied, and various numerical methods are proposed [18]. When A is large and sparse, the power method is useful and simple to evaluate the maximal norm eigenvalue and associated eigenvector. For a quaternion matrix A, since the non-commutativity of quaternion multiplication, there is no very close relation between left and right eigenvalues [1]. Compared with the left eigenvalue problem, the right eigenvalue problem has been well studied, since they are invariant under the similarity transformation, right eigenvalues are more useful [1]. For example, the quaternion Schrödinger equation plays an important role in quaternion mechanics, and the study of the quaternion Schrödinger equation $\frac{\partial}{\partial t}|f>=-A|f$ is reduced to the study of quaternion eigen-problem $A\alpha=\alpha\lambda$ with A a quaternion matrix, and the computation of the principle right eigenvalue (i.e., the right eigenvalue with maximal norm) is a key problem to solve quaternion Schrödinger equation [2,19]. If A is a quaternion Hermitian matrix, then A is unitary similarity to a real diagonal matrix, thus the methods for solving real or complex matrices eigenvalue problem can be directly extended to solve quaternion Hermitian eigenvalue problem. On the contrary, if A is non-Hermitian, the situation is quite different and more complicated, the computational methods of real or complex matrix eigenvalue problem cannot be directly extended to the quaternion non-Hermitian matrix, and the working on computing right eigenvalues of quaternion matrices is much more difficult than that of real or complex matrices.

To our knowledge, there is no paper discussing the power method for quaternion matrix. In this paper, our goal is to propose the power method for quaternion right eigenvalue problem.

In this paper, we use the following notation. $\mathbb{R}$ and $\mathbb{Q}$ denote real number field and quaternion skew-field, respectively. $\mathbb{F}^{m\times n}$ denotes the set of all $m\times n$ matrices on $\mathbb{F}$. For any matrix $A\in\mathbb{F}^{m\times n}$, A^T, $\bar{A}$ and A^H present the transpose, conjugate and conjugate transpose of A, respectively.

This paper is organized as follows. In Section 2, we recall some preliminary results about right eigenvalue problem used in the paper. In Section 3.1, we present the power method of right eigenvalue problem for quaternion Hermitian matrices. In Section 3.2, we present the power method of right eigenvalue problem for quaternion non-Hermitian matrices. In Section 4, we give numerical examples. Finally in Section 5, we make some concluding remarks.

2. Preliminaries

In this section, we recall some basic properties about quaternion, quaternion matrices and quaternion right eigenvalue problem. See, e.g.,[1].

A quaternion $q\in\mathbb{Q}$ is expressed as

$$q=a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k},$$

in which $a,b,c,d\in\mathbb{R}$, and three imaginary units $\mathbf{i},\mathbf{j},\mathbf{k}$ satisfy

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1,\ \mathbf{ij}=\mathbf{k},\ \mathbf{jk}=\mathbf{i}\ \textit{and}\ \mathbf{ki}=\mathbf{j}.$$

The quaternion skew-field $\mathbb{Q}$ is an associative but non-commutative algebra of rank four over $\mathbb{R}$, endowed with an involutory antiautomorphism

$$q\to\overline{q}=a-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}.$$

Every non-zero quaternion is invertible, and the unique inverse is given by $q^{-1}=\overline{q}/|q|^2$, in which the quaternion norm $|q|$ is defined by

$$|q|^2=\overline{q}q=a^2+b^2+c^2+d^2.$$

Two quaternions x and y are said to be similar if there exists a nonzero quaternion u such that $u^{-1}xu=y$; this is written as $x\sim y$. Obviously, x and y are similar if and only if there is a unit quaternion v such that $v^{-1}xv=y$, and two similar quaternions have the same norm. It is routine to check that $\sim$ is an equivalence relation on the quaternion. We denote by $[x]$ the equivalence class containing x.

Lemma 2.1. *If $q=q_1+q_2\mathbf{i}+q_3\mathbf{j}+q_4\mathbf{k}$ with $q_1,q_2,q_3,q_4\in\mathbb{R}$, then q and $q_1+\sqrt{q_2^2+q_3^2+q_4^2}\mathbf{i}$ are similar, namely, $q\in[q_1+\sqrt{q_2^2+q_3^2+q_4^2}\mathbf{i}]$.*

In fact, if $q_3^2+q_4^2\neq 0$, we choose $x=(\sqrt{q_2^2+q_3^2+q_4^2}+q_2)-q_4\mathbf{j}+q_3\mathbf{k}$ and $u_\lambda=\frac{x}{|x|}$, then $x^{-1}qx=\overline{u}_\lambda qu_\lambda=x(q_1+\sqrt{q_2^2+q_3^2+q_4^2}\mathbf{i})$.

We can define right linear independence over $\mathbb{Q}$ for a set of quaternion vectors.

Definition 2.1. For vectors $\eta_1,\eta_2,\ldots,\eta_r\in\mathbb{Q}^n$, we say that $\eta_1,\eta_2,\ldots,\eta_r$ are right linearly dependent,

$$\eta_1k_1+\eta_2k_2+\cdots+\eta_rk_r=0,\ \textit{implies}\ k_1=k_2=\cdots=k_r=0.$$

Otherwise, we say that $\eta_1,\eta_2,\ldots,\eta_r$ are right linearly dependent.