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Internality of generalized averaged Gaussian quadrature rules and truncated variants for modified Chebyshev measures of the second kind

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ABSTRACT

Generalized averaged Gaussian quadrature rules associated with some measure, and truncated variants of these rules, can be used to estimate the error in Gaussian quadrature rules. However, the former quadrature rules may have nodes outside the interval of integration and, therefore, it may not be possible to apply them when the integrand is defined on the interval of integration only. This paper investigates whether generalized averaged Gaussian quadrature rules associated with modified Chebyshev measures of the second kind, and truncated variants of these rules, are internal, i.e. if all nodes of these quadrature rules are in the interval of integration.

1. Introduction

Let p_k denote the monic polynomial of degree k that is orthogonal to $\mathbb{P}_{k-1}$ (the set of all polynomials of degree less than or equal to $k-1$) with respect to a nonnegative measure $d\lambda$ supported on an interval $[a, b]$, i.e.

$$\int_a^b t^j p_k(t)\, d\lambda(t) = 0, \quad j = 0, 1, \ldots, k-1.$$

Recall that the polynomials p_k satisfy a three-term recurrence relation of the form

$$p_{k+1}(t) = (t - \alpha_k)p_k(t) - \beta_k p_{k-1}(t), \quad k = 0, 1, \ldots, \tag{1.1}$$

where $p_{-1}(t) := 0$, $p_0(t) := 1$ and the coefficients β_k are positive. It is well known that the unique interpolatory quadrature rule with n nodes and the highest possible algebraic degree of precision, $2n-1$, is the Gaussian rule with respect to the

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measure $d\lambda$,

$$G_n[f] = \sum_{j=1}^{n} \omega_j^G f\left(x_j^G\right), \tag{1.2}$$

i. e.

$$G_n[f] = \int_a^b f(t)d\lambda(t) \qquad \forall f \in \mathbb{P}_{2n-1};$$

see, e.g., Gautschi [1] or Szegő [2]. Therefore, Gauss rules (1.2) are commonly used to approximate integrals

$$I[f] := \int_a^b f(t)d\lambda(t).$$

It is an important problem in computational mathematics to find another quadrature rule Q, that is of higher algebraic degree of precision than $G_n[f]$, for estimating the error $|I[f] - G_n[f]|$ by calculating

$$|Q[f] - G_n[f]|.$$

For instance, Q may be chosen as the $(2n+1)$-node Gauss–Kronrod quadrature rule, H_{2n+1}, of degree of precision at least $3n+1$, when it exists. However, Gauss-Kronrod rules are known not to exist for several of the classical weight functions, including for the Hermite and Laguerre weight functions; see [3–5]. A nice recent survey of Gauss–Kronrod rules is provided by Notaris [6].

The non-existence of Gauss–Kronrod rules inspired Laurie [7] to develop anti-Gaussian quadrature rules. These rules always exist and have real nodes, at most two of which are outside the interval of integration. Moreover, all quadrature weights are positive and the anti-Gaussian rules easily can be constructed.

Spalević [8] (by following Peherstorfer [9]; see also [10]) proposed a simple numerical method for constructing generalized averaged Gaussian formulas $\widehat{G}_{2n+1}$. They are based on the zeros of the polynomial

$$t_{2n+1} := p_n \cdot F_{n+1},$$

where the polynomial F_{n+1} is defined by

$$F_{n+1} := p_{n+1} - \bar{\beta}_{n+1} \cdot p_{n-1} \tag{1.3}$$

for a suitable coefficient $\bar{\beta}_{n+1} > 0$. It is shown in [8] that $\widehat{G}_{2n+1}$ has algebraic degree of precision $2n+2$ when $\bar{\beta}_{n+1} = \beta_{n+1}$, where β_{n+1} is defined in (1.1). In this case, we denote the quadrature rule by $\widehat{G}^S_{2n+1}$. Details on these rules and discussions on applications can be found in the recent papers [8,11–15].

When instead $\bar{\beta}_{n+1} = \beta_n$ in (1.3), the averaged Gaussian formula $\widehat{G}_{2n+1}$ becomes the average Gaussian quadrature rule $\widehat{G}^L_{2n+1}$ introduced by Laurie [7]. It has algebraic degree of precision $2n+1$.

Truncated versions of the quadrature formula $\widehat{G}^S_{2n+1}$ with $2n-r+1$ nodes and with the same algebraic degree of precision as $\widehat{G}^S_{2n+1}$ have been considered in [11,14]. We denote the truncated rules by $Q^{(n-r)}_{2n-r+1}$ $(n \geq 2)$, $r = 1, 2, \ldots, n-1$. The simplest truncated generalized averaged Gaussian quadrature formula is

$$Q^{(1)}_{n+2}[f] = \sum_{j=1}^{n+2} \omega_j^T f\left(\tau_j^T\right);$$

see [11, Eq. (4.1)] for discussions. Its nodes are the zeros of the polynomial

$$t_{n+2}(t) := (t - \alpha_{n-1})p_{n+1}(t) - \beta_{n+1}p_n(t). \tag{1.4}$$

It is well known that the nodes of the Gaussian rule (1.2) live in the open interval (a, b); see Gautschi [1] or Szegő [2]. Therefore, Gaussian rules can be applied when the integrand f is defined on the interval $[a, b]$. However, the quadrature rules $\widehat{G}^S_{2n+1}$, $\widehat{G}^L_{2n+1}$, and truncated versions $Q^{(n-r)}_{2n-r+1}$ of the rules $\widehat{G}^S_{2n+1}$ may have nodes outside this interval. In this case, they can be applied to estimate the error in $G_n[f]$ only when the integrand f is defined in a sufficiently large interval that contains $[a, b]$.

We are interested in whether for special measures defined below, the nodes of the rules $\widehat{G}^S_{2n+1}$, $\widehat{G}^L_{2n+1}$, and $Q^{(n-r)}_{2n-r+1}$ are internal, i. e. whether all nodes live in $[a, b]$. When this is the case, they can be applied to estimate the error in the Gauss rules (1.2) for all integrands defined in this interval.

The internality of the quadrature rules $\widehat{G}^L_{2n+1}$, $\widehat{G}^S_{2n+1}$, and $Q^{(1)}_{n+2}$ $(n \geq 2)$ for the classical weight functions is investigated in [7,8], and [11], respectively. For Bernstein–Szegő weight functions the internality of these quadrature rules is discussed in [16]. The latter weight function were introduced by Gautschi and Notaris [17], who studied the internality of Gauss-Kronrod rules for this weight function. The present paper extends the investigations in [7,8], and [11], to modified Chebyshev measures. These measures have been considered by Milovanović et al. [18], who established expressions in closed form for