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A general framework for the optimal approximation of circular arcs by parametric polynomial curves

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ABSTRACT

We propose a general framework for a geometric approximation of circular arcs by parametric polynomial curves. The approach is based on a constrained uniform approximation of an error function by scalar polynomials. The system of nonlinear equations for the unknown control points of the approximating polynomial given in the Bézier form is derived and a detailed analysis provided for some low degree cases which were not studied yet. At least for these cases the solutions can be, in principle, written in a closed form, and provide the best known approximants according to the simplified radial distance. A general conjecture on the optimality of the solution is stated and several numerical examples conforming theoretical results are given.

1. Introduction

Circular arcs are one of the fundamental geometric primitives and together with straight lines they have been one of the cornerstones of several graphical and control systems. Later on parametric polynomial representations of geometric objects have been widely used in applications and successfully upgraded to non-uniform rational basis splines (NURBS) which nowadays provide an intuitive approach towards the construction and modeling of curves and surfaces used in computer aided geometric design (CAGD) and related fields. However, there is still an interest in parametric polynomial curves, since they provide even more simple representations of geometric objects and might still be in use in some software standards. On the other hand, optimal approximation of special classes of functions or parametric objects by polynomials has always been a theoretical issue (Chebyshev alternation theorem [1], Stone–Weierstrass approximation theorem [2], etc.). Circular arcs form one such class of curves, since it is well known that a circular arc of positive length cannot be exactly represented in a polynomial form.

A common way to construct parametric polynomial approximants of a circular arc is to interpolate corresponding geometric quantities. This usually includes interpolation of boundary points, corresponding tangent directions, signed curvatures, etc. The result is so called geometric parametric polynomial approximants (G^k approximants), which can be put together to geometrically smooth spline curves.

When we are dealing with approximations, the fundamental question is a measure of a distance between a parametric polynomial approximant and a circular arc. One of the standard measures in this case is the radial distance measuring the

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distance of the point on the parametric polynomial to the corresponding point on the circular arc in the radial direction. It can be shown that under some additional assumptions it coincides with the well known Hausdorff distance [3,4]. It is more common to use a simplified version of the radial distance, the difference between the square of the distance of the point on the parametric polynomial curve to the center of the circular arc and the square of its radius. The later one is more attractive since it simplifies the analysis of the existence and uniqueness of the approximant but still preserves the optimality of the approximation order. However, it is important to emphasize that the optimal solutions according to these two measures do not coincide in general.

The list of literature dealing with different types of geometric approximants of a circular arc is long and we shall mention just the most relevant references according to our approach described later. Parabolic G^0 interpolants were considered in [5]. This is actually one of only a few cases where the optimality of the solution was proved. Different types of G^1 and G^2 cubic geometric interpolants were given in early papers [6] and [7]. Several types of quartic and quintic Bézier curves were considered in [3], and deeper analysis of some geometric quintic approximants can be found in [8]. Many new cubic and quartic approximants were also proposed in [9–13]. However, in none of the above papers the optimality of the solution has been considered. The paper [14] is beside [5] the only one where optimality of some approximants was formally shown. The authors managed to prove it for cubic G^1 and quartic G^2 approximants.

Some authors also considered the approximation of circular arcs by general degree parametric polynomials. In [15], the Taylor type geometric interpolation, i.e., interpolation at just one point was considered for all odd degree polynomials. For even degree ones the results can be found in [16] and in a more general form in [17]. The approximation of the whole circle by Lagrange type approximants can be found in [18] and in [4].

The aim of this paper is to present a general framework providing optimal geometric approximants for general degree n of the parametric polynomial and for any order k of geometric smoothness. The idea relies on the constrained uniform approximation of the error function by scalar polynomials.

The paper is organized as follows. In Section 2 the problem is explained in detail and the radial distance and its simplification are precisely introduced. A general conjecture that the proposed approach provides optimal solutions is stated. Section 3 concerns a constrained uniform approximation of an error function by scalar polynomials. A general theory is briefly revised and some particular cases needed later are carefully analyzed. Next section describes optimal G^k approximation of circular arcs by parametric polynomial curves. In particular, it provides the system of nonlinear equations which has to be solved. In Section 5 some particular cases are studied in detail. For some of them the optimality is reconfirmed, but for all of them it is shown that they provide the minimal simplified radial distance among all known approximants. In the last section some concluding remarks and suggestions for possible future research are given.

2. Preliminaries

We shall consider the following problem. Let $\boldsymbol{c} : [-\varphi, \varphi] \to \mathbb{R}^2$, $0 < \varphi \leq \pi/2$ be a standard nonpolynomial parameterization of a circular arc. Due to simple affine transformations it is enough to consider the unit circular arcs only, centered at the origin and symmetric with respect to the first coordinate axis. Thus we can assume that $\boldsymbol{c}(s) = (\cos s, \sin s)^T$. Our goal is to find as good as possible approximation of $\boldsymbol{c}$ by parametric polynomial curve $\boldsymbol{p}_n : [-1, 1] \to \mathbb{R}^2$ of degree $n \in \mathbb{N}$. It is convenient to express $\boldsymbol{p}_n = (x_n, y_n)^T$, where x_n and y_n are polynomials of degree at most n, in Bézier form, i.e.,

$$\boldsymbol{p}_n(t) = \sum_{j=0}^{n} B_j^n(t)\boldsymbol{b}_j, \tag{1}$$

where $B_j^n, j = 0, 1, \ldots, n$, are (reparameterized) Bernstein polynomials over $[-1, 1]$, given as

$$B_j^n(t) = \binom{n}{j}\left(\frac{1+t}{2}\right)^j\left(\frac{1-t}{2}\right)^{n-j},$$

and $\boldsymbol{b}_j \in \mathbb{R}^2, j = 0, 1, \ldots, n$, are the control points.

The quality of the approximation will be measured by radial distance. For each point on the parametric curve $\boldsymbol{p}_n$ the closest point on the circular arc $\boldsymbol{c}$ in the radial direction will be considered. In general, it might happen that no such point exists on $\boldsymbol{c}$, but some further restrictions on $\boldsymbol{p}_n$ will override this problem. The formal definition of the radial distance $\widetilde{\psi}_n$ is

$$\widetilde{\psi}_n : [-1, 1] \to [0, \infty), \quad \widetilde{\psi}_n(t) := \left|\sqrt{x_n(t)^2 + y_n(t)^2} - 1\right| = |\|\boldsymbol{p}_n(t)\|_2 - 1|,$$

where $\|\cdot\|_2$ is the standard Euclidean norm on $\mathbb{R}^2$. Function $\widetilde{\psi}_n$ is an upper bound for the parametric distance d_P, studied in detail in [15]. For $\boldsymbol{c}$ and $\boldsymbol{p}_n$ it is defined as

$$d_P(\boldsymbol{c}, \boldsymbol{p}_n) = \inf_{\rho} \max_{t\in[-1,1]} \|(\boldsymbol{c} \circ \rho)(t) - \boldsymbol{p}_n(t)\|_2,$$

where $\rho : [-1, 1] \to [-\varphi, \varphi]$ is a smooth bijection for which $\rho' > 0$. Clearly, d_P is in general an upper bound for the well known Hausdorff distance d_H. If the radial distance between $\boldsymbol{c}$ and $\boldsymbol{p}_n$ is well defined, it can be shown that actually

$$d_H(\boldsymbol{c}, \boldsymbol{b}) = d_P(\boldsymbol{c}, \boldsymbol{p}_n) = \max_{t\in[-1,1]} \widetilde{\psi}_n(t)$$