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Adaptive Monte Carlo methods for solving hyperbolic telegraph equation

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ABSTRACT

Here, empirical studies with adaptive Monte Carlo and the almost optimal Monte Carlo methods for solving hyperbolic telegraph equations are discussed. Due to applying the difference discretization, a large sparse system is obtained. Adaptive Monte Carlo methods are employed to solve this large system. Illustrative examples of telegraph equation are considered to compare the power and accuracy of adaptive Monte Carlo methods. Numerical test results show that the almost optimal Monte Carlo method has much slower convergence speed, adaptive importance sampling method does not always converge faster than adaptive correlated sequential sampling method, and the advantage of adaptive correlated sequential sampling method over adaptive importance sampling and the almost optimal Monte Carlo methods, as well as some deterministic methods, is evident.

1. Introduction

This article focuses on studying the numerical solutions of the second-order one-space-dimensional hyperbolic telegraph type equation. An infinitesimal piece of telegraph wire as an electrical circuit can be modeled by telegraph equation models which describe the voltage and current in a double conductor with distance x and time t [1]. The standard form of the second-order telegraph equation is

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x, t)\,, \; a \le x \le b\,, \; 0 \le t \le T, \tag{1}$$

with initial conditions

$$u(x, 0) = \varphi_1(x), \;\; \frac{\partial u}{\partial t}(x, 0) = \varphi_2(x), \;\; a \le x \le b, \tag{2}$$

and boundary conditions

$$u(a, t) = \psi_1(t), \;\; u(b, t) = \psi_2(t), \;\; 0 \le t \le T, \tag{3}$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2, f$ are all known functions, α, β are given real constant coefficients and $u(x, t)$ is unknown and describes the voltage and current in a double conductor with distance x and time t. For $\alpha > 0, \beta = 0$, Eq. (1) describes a damped wave equation, and if $\alpha > \beta > 0$, it represents a telegraph equation. Eq. (1) is the basis for fundamental equations of atomic physics and is commonly used to describe the vibrations of structures (e.g. buildings and machines).

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Recently, considerable attention has been given to the numerical solution of the hyperbolic telegraph equations due to their applications in modeling many physical phenomena such as vibrations of structures (e.g., building, beams and machines) and atomic physics [2]. In recent years, numerical methods in the literature have been developed for solving Eq. (1) by using techniques such as collocation method based on redefined extended cubic B-spline basis function and Quartic B-spline collocation methods [3,4], Differential Quadrature method [5], Modified cubic B-spline collocation method [6], the finite difference scheme and the DGJ method [7,8], Chebyshev spectral collocation method [9], polynomial scaling function [10] and the Bessel function [11]. The above mentioned methods were referred to deterministic methods.

In general, a matrix equation should be obtained when numerical methods were applied to Eq.(1) such as collocation method, finite difference method, polynomial scaling function method and Bessel function method. For the gained matrix equation, if the order of coefficient matrix is large and coefficient matrix is diagonally dominant, such as those arising from approximations of partial differential equations [12]. Monte Carlo (say MC) method has advantages over deterministic methods in some situations. There were some basic advantages of MC method. It is good for diagonally dominant systems in which convergence is rapid. However, when the matrix is dense, adaptive MC methods are more efficient than the almost optimal MC (say AOMC) method. MC method can be used to evaluate only one or some of the components of the solution vector which are needed, whereas deterministic methods generally cannot do so [13].

An introduction and some practical recommendations of MC method and its application in solving the matrix equation can be found in [12,14,15]. Halton [16,17] was the first to use adaptive MC methods, both the correlated sequential sampling and the importance sampling methods, to speed up the convergence in estimating solutions of systems of algebraic equations. Lai [18] discussed empirical studies with both adaptive correlated sequential sampling (say ACSS) and adaptive importance sampling (say AIS) methods which can be used in solving the matrix equations. However, there is little about comparison among AOMC, AIS and ACSS methods for numerically solving the second-order one-dimensional hyperbolic telegraph equations, especially comparison between AIS method and ACSS method. The empirical studies showed that AIS method is more efficient than ACSS method in [18]. At the moment, our numerical results indicate that ACSS method has the same accuracy (if the entries of matrix are all nonnegative) or higher accuracy (if some of the entries are negative) and lower computational costs than AIS method. In addition, the CPU times used on a normal computer to obtain the solutions of linear algebraic system are compared.

The remainder of this article is organized as follows: In Section 2, the problems (1)–(3) are discretized by compact finite difference approximation in the directions of time and space, which produces a system of algebraic equation. The complexity of MC method for solving linear systems is introduced in Section 3. In Section 4, adaptive MC and the almost optimal MC methods are introduced to solve the linear system. Section 5 discusses error analysis of MC methods. In Section 6, numerical experiments are conducted to demonstrate the efficiency of the proposed method computationally for Eq. (1).

2. Temporal and spatial discretization

Using the compact finite difference scheme [19] with the time step $\triangle t=\tau=\frac{T}{n}$ and the space step $\triangle x=h=\frac{b-a}{m}$, we discretized the time derivative and the space derivative terms of the given Eqs. (1)–(3) and have

$$\begin{aligned}&\frac{1}{12}\left(\delta_t^2u_{p-1}^q+10\delta_t^2u_p^q+\delta_t^2u_{p+1}^q\right)+\frac{2\alpha}{12}\left(\delta_tu_{p-1}^q+10\delta_tu_p^q+\delta_tu_{p+1}^q\right)\\&\quad+\frac{\beta^2}{12}\left(u_{p-1}^q+10u_p^q+u_{p+1}^q\right)=\frac{1}{2}\left(\delta_x^2u_p^{q-1}+\delta_x^2u_p^{q+1}\right)\\&\quad+\frac{1}{12}\left(f_{p-1}^q+10f_p^q+f_{p+1}^q\right),\ \ 1\le p\le m-1,\ \ 1\le q\le n-1,\end{aligned}\tag{4}$$

with initial and the boundary conditions

$$u_p^0=\varphi_1(x_p),\ \ 0\le p\le m,\tag{5}$$

$$u_p^1=\varphi_1(x_p)+\tau\varphi_2(x_p)+\frac{\tau^2}{2}\left(\varphi_1''(x_p)-2\alpha\varphi_2(x_p)-\beta^2\varphi_1(x_p)+f(x_p,0)\right)\tag{6}$$

$$\begin{aligned}u_0^{2l}&=\psi_1(t_{2l}),\ u_m^{2l}=\psi_2(t_{2l}),\ \ 2\le 2l\le n,\\u_0^{2l+1}&=\psi_1(t_{2l+1})+u_0^1-\psi_1(t_1),\ u_m^{2l+1}=\psi_2(t_{2l+1})+u_m^1-\psi_2(t_1),\\&\quad 3\le 2l+1\le n\end{aligned}\tag{7}$$

where $u_p^q=u(ph,q\tau)$, $\delta_t^2u_{p-1}^q=\frac{u_{p-1}^{q+1}-2u_{p-1}^q+u_{p-1}^{q-1}}{\tau^2}$, $\delta_tu_{p-1}^q=\frac{u_{p-1}^{q+1}-u_{p-1}^{q-1}}{2\tau}$, $\delta_x^2u_p^{q-1}=\frac{u_{p+1}^{q-1}-2u_p^{q-1}+u_{p-1}^{q-1}}{h^2}$.

By sorting and merging (4)–(7), we get the following matrix equation:

$$Au^{q+1}=b,\tag{8}$$

here $u^{q+1}=(u_1^{q+1},u_2^{q+1},\ldots,u_{m-1}^{q+1})^{\mathrm{T}}$ is unknown and going to be found by MC methods.

Matrix equation (8) can be converted to the below form:

$$u^{q+1}=Lu^{q+1}+b_1,\tag{9}$$