Journal of Computational and Applied Mathematics 343 (2018) 635–642

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A new integral formula for the variation of matrix elastic energy of heterogeneous materials

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ARTICLE INFO

Article history:
Received 26 September 2017
Received in revised form 7 January 2018

Keywords:
Integral formula
Elastic energy
Heterogeneous materials
Boundary element method

ABSTRACT

This work presents a new integral formula for the variation of matrix elastic energy caused by the inclusion, which only contains the displacements on the interface between inclusion and matrix. Compared with the existing formula, the present formula avoids the corner point problems in the implementation of the boundary element method (BEM) so that it can conveniently deal with the complex shape inclusion problems. In numerical calculation, 3-node (8-node) quadratic boundary elements for two (three) dimensional problems are used to discretize the interface between inclusion and matrix. Numerical results are compared with the analytical solutions available.

1. Introduction

Composite materials with various inclusions have become increasingly important to improve their mechanical behavior. In this paper, the present study focuses on the effect of inclusion on the variation of matrix elastic energy.

Christensen [1] presented an integral formula for the variation of matrix elastic energy due to the existence of inclusion, in which the displacements and tractions on the interface between inclusion and matrix are contained. Based on this formula, the variation of matrix elastic energy caused by the inclusion can easily be calculated using the BEM [2], which only uses the displacements and tractions on the inclusion–matrix interface. However, for complex shape inclusions, the discontinuous boundary elements should be adopted near the corners since there are discontinuous tractions at the corners. Therefore, it is not convenient to carry out the numerical implementation of the BEM, especially for 3D heterogeneous materials. Similar case also appears in the calculation of the variation of matrix heat energy caused by the inclusion for steady state thermal conductivities [3].

Motivated by an integral formula for steady state heterogeneous materials [3], which only contains the temperatures on the inclusion–matrix interface, a new integral formula for calculating the variation of matrix elastic energy caused by the inclusion, which only contains the displacements on the inclusion–matrix interface, is proposed. The present formula is especially suitable for investigating the variation of the elastic energy for heterogeneous materials. The distinct advantage is that it does not contain the tractions on the inclusion–matrix interface so that the corner point problems of heterogeneous materials can be avoided.

2. Basic formula

One inclusion of another material is embedded into one homogeneous media subjected to the remote loading. The variation of matrix elastic energy due to the existence of inclusion has the following form [1]:

$$\Delta U = \frac{1}{2}\int_{\Gamma}\left(t_i^0 u_i - t_i u_i^0\right) d\Gamma \tag{1}$$

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https://doi.org/10.1016/j.cam.2018.04.064
0377-0427/

where u_i and t_i ($i = 1, 2, 3$ for 3D problems) are, respectively, the ith displacement and ith traction components over the interface Γ between inclusion and matrix. The repeated indices imply summation. $t_i = \sigma_{ij} n_j$ in which σ_{ij} is the stress tensor, while n_i is the ith direction cosine of unit vector $\mathbf{n}$ relative to the existing coordinate system. The symbols with the superscript 0 denote the variables that are generated by the remote loading on the interface Γ between matrix and inclusion with the same material as the matrix. $t_i^0 = \sigma_{ij}^0 n_j$ is easily obtained using the elasticity relations. The formula is considered to be of great advantage due to its interface integral.

The formula (1) can be calculated using the BEM [2]. However, the special technique should be adopted for the inclusions with irregular shapes, e.g. discontinuous elements are used to overcome the corner point problems. In order to avoid the use of discontinuous elements, the formula (1) can be further improved. The detailed derivations are as follows:

The second term in the right hand side of Eq. (1) can be rewritten as

$$\int_\Gamma t_i u_i^0 d\Gamma = \int_\Omega \sigma_{ij} \varepsilon_{ij}^0 d\Omega \tag{2}$$

where the strain tensor $\varepsilon_{ij}^0 = \frac{1}{2}\left(u_{i,j}^0 + u_{j,i}^0\right)$ in which the tensor $u_{i,j}^0$ is called the displacement gradient tensor. In derivation process of Eq. (2), the equilibrium equation $(\sigma_{ij,j} = 0)$ and Green's theorem have been used.

The general 3D constitutive law for linear elastic materials can be expressed in standard tensor notation by the following form [4]:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \tag{3}$$

where C_{ijkl} is a four order elasticity tensor, and for the isotropic materials, it can be given in the form below:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{4}$$

where $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and is called Lame's constant, and $G = \frac{E}{2(1+\nu)}$ is the shear modulus in which E is called the modulus of elasticity, and ν is referred to as Poisson's ratio.

It is assumed that the inclusion and matrix have the same Poisson's ratio. By means of Eq. (4), we have the following form:

$$\frac{C_{ijkl}^I}{E^I} = \frac{C_{ijkl}^M}{E^M} \tag{5}$$

where the superscripts I and M denote the inclusion and matrix, respectively. Eq. (5) can be rewritten as

$$C_{ijkl}^I = \frac{E^I}{E^M} C_{ijkl}^M. \tag{6}$$

Substituting Eq. (6) into Eq. (3) yields

$$\sigma_{ij}^I = \frac{E^I}{E^M} C_{ijkl}^M \varepsilon_{kl}. \tag{7}$$

Thus, the integral in the right hand side of Eq. (2) becomes

$$\int_\Omega \sigma_{ij} \varepsilon_{ij}^0 d\Omega = \int_\Omega \frac{E^I}{E^M} C_{ijkl}^M \varepsilon_{kl} \varepsilon_{ij}^0 d\Omega = \frac{E^I}{E^M} \int_\Omega \varepsilon_{kl} \sigma_{kl}^0 d\Omega = \frac{E^I}{E^M} \int_\Gamma t_k^0 u_k d\Omega. \tag{8}$$

The final form of Eq. (8) is obtained by using the equilibrium equation $(\sigma_{ij,j}^0 = 0)$ and Green's theorem. Substituting Eq. (8) back into Eq. (1) produces one simplified formula of the variation of matrix elastic energy for heterogeneous material as follows:

$$\Delta U = \frac{1}{2}\left(1 - \frac{E^I}{E^M}\right) \int_\Gamma t_i^0 u_i \, d\Gamma. \tag{9}$$

Compared L with Eq. (1), Eq. (9) only contains the displacements over the interface between inclusion and matrix, which can be conveniently calculated using the BEM [3]. Eq. (9) can easily be generalized to multiple inclusions, i.e.

$$\Delta U = \frac{1}{2} \sum_{I=1}^{N} \left(1 - \frac{E^I}{E^M}\right) \int_{\Gamma_I} t_i^0 u_i \, d\Gamma \tag{10}$$

where N expresses the number of inclusions. Here, I denotes the Ith inclusion. When $E^I = E^M$, i.e. no inclusions, ΔU is equal to zero as expected.

In order to calculate the variation of matrix elastic energy of heterogeneous material, the displacements (no tractions) over the interface Γ between inclusion and matrix must first be obtained. As well known, the BEM will be a good choice for calculating the interface displacements, i.e. the interface integral equation for heterogeneous material is as follows [5]:

$$c_{ij}\left(1 + \frac{E^I}{E^M}\right) u_j(p) = u_i^0(p) - \int_{\Gamma_I} \left(1 - \frac{E^I}{E^M}\right) T_{ij}(p, q)\, u_j(q)\, d\Gamma \tag{11}$$