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Semismooth Newton methods with domain decomposition for American options

Hong-Jie Zhao, Haijian Yang *

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China

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ABSTRACT

In this paper, we develop a class of parallel semismooth Newton algorithms for the numerical solution of the American option under the Black–Scholes–Merton pricing framework. In the approach, a nonlinear function is used to transform the complementarity problem, which arises from the discretization of the pricing model, into a nonlinear system. Then, a generalized Newton method with a domain decomposition type preconditioner is applied to solve this nonlinear system. In addition, an adaptive time stepping technique, which adjusts the time step size according to the initial residual of Newton iterations, is applied to improve the performance of the proposed method. Numerical experiments show that the proposed semismooth method has a good accuracy and scalability.

1. Introduction

The model of the American option can be posed as a linear complementarity problem and various methods have been invested to accelerate the simulation [1–4]. For all the numerical methods, the main challenge to the simulation of the American option is how to solve the large system of equations. We briefly mention a few related publications that partially motivated our current work. The Ref. [5] established the American put option model and then presented a simple numerical procedure to obtain the approximate solution. The projected successive over relaxation (PSOR) method, produced by Crywer [6], is another popular approach for the solution of the linear complementarity problem. This method for solving the American option is described in detail in book [7], but the convergence rate of the PSOR method is not well when using finer space mesh. In [8], Forsyth and Vetzal proposed a penalty method for the American put option model and studied the convergence rates of this method. The penalty method is adding a small, continuous penalty term to the Black–Scholes equation and then the solution is obtained with a generalized Newton method. The Ref. [9] used the front-fixing scheme to solve the American option by transforming the model problem into a nonlinear parabolic differential equation. In [10], Ikonen and Toivanen gave the operator splitting methods for the linear complementarity problem arising from the American option. These papers investigate many interesting aspects of the solution of the American option, but parallel computing is not their main concern. In this paper, we focus on the class of semismooth Newton methods and some highly parallel domain decomposition methods for the American option.

The semismooth Newton method is an efficient and popular technique for solving the semismooth nonlinear problems [11–13]. When combined with Krylov–Schwarz method, the semismooth Newton–Krylov–Schwarz framework provides a powerful technique for the solution of the resultant nonlinear system [14–16]. More precisely, in the approach, (a) an NCP function is used to transform the linear complementarity problem into a nonlinear system; (b) a generalized Newton

* Corresponding author.
E-mail addresses: zhaohongjie@hnu.edu.cn (H.-J. Zhao), haijianyang@hnu.edu.cn (H. Yang).

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method is employed to solve the nonlinear system; (c) the linear system at each Newton process is solved by a Krylov subspace iteration method; (d) the class of domain decomposition preconditioners is employed to accelerate the Krylov iterations. The success of the overall approach depends heavily on the preconditioner. Thus, in the study, we choose the overlapping additive Schwarz type domain decomposition methods [17,18] to build the preconditioner, since it is able to substantially reduce the condition number of the linear system and meanwhile provides the scalability for massively parallel computing. In addition to that, for the long-term simulation of the American option, the evolution of the system usually admits various time scales and the calculation often lasts for a long time. Therefore, it is necessary to use an adaptive time step control for the numerical simulation. Based on this, we present an implicit method with adaptive time stepping, where the time step size is chosen by comparing the change of the solution between two subsequent time steps. Our experiments show that these strategies provide a good improvement of the overall method in terms of the total computing time and the number of linear iterations.

The rest of the paper is organized as follows. In Section 2, we present an implicit finite difference scheme for the American put option. Section 3 presents the semismooth algorithms and the constructs of the three kinds of additive Schwarz preconditioners. Some numerical experiments are provided in Section 4 to validate the proposed algorithm and study the parallel performance. Finally, the paper is ended with conclusions in Section 5.

2. The model and its discretization

The pricing model of the American put option can be expressed based on the Black–Scholes partial differential equation [19]

$$\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r-q) S \frac{\partial V}{\partial S} + r V = 0, \tag{1}$$

where V is the option value, S is the asset price, τ is the time to maturity, q is the constant continuous dividend yield, σ is the constant volatility, and r is the constant risk free interest rate. Meanwhile, the boundary and initial conditions associated to (1) are

$$\begin{cases} V(S,0) = \max(E-S,0), \\ V(0,\tau) = E, \quad \tau \in [0,T], \\ V(S,\tau) = 0, \quad \text{as } S \to \infty\,,\ \tau \in [0,T] \end{cases} \tag{2}$$

where T denotes the expiration date and E is the strike price.

Because the American option embeds an early exercise right when compared with the European one, there must have a constraint to avoid arbitrage possibilities

$$V(S,\tau) \ge \max(E-S,0) := \Phi(S),$$

where Φ is the payoff function of the option contract. Then the most important part of the pricing model is the introduction of a free boundary. It is worth noting that the free boundary does not appear in the final expression. The free boundary is the optimal exercise boundary and it separates the continuation region and stopping region, this is seen more clearly in Fig. 1. As shown in the figure, S^* is the optimal exercise price which forms the exercise boundary and it can be interpreted as the maximum value of S for which satisfies

$$V(S,\tau) - \max(E-S,0) = 0. \tag{3}$$

If the asset price S falls within the stopping region, it is better to exercise the option. The option value V is equal to the payoff function and does not satisfy the Black–Scholes equation because of $0 < S \le S^*(\tau) < S^*(0^+) = E \cdot \min(1, r/q)$. So we have

$$\begin{cases} V(S,\tau) = \max(E-S,0) = E-S, \\ \left(\frac{\partial}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2} - (r-q) S \frac{\partial}{\partial S} + r\right)(E-S) = rE - qS > 0. \end{cases} \tag{4}$$

While in the continuation region, continuing hold is chosen. There is $S > S^*(\tau)$, the option value V is greater than the payoff, and V satisfies the Black–Scholes equation. These lead to another formulation

$$\begin{cases} V(S,\tau) > \max(E-S,0), \\ \frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r-q) S \frac{\partial V}{\partial S} + r V = 0. \end{cases} \tag{5}$$