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On developing a stable and quadratic convergent method for solving absolute value equation

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ABSTRACT

We modify the generalized Newton method, proposed by Mangasarian (2007), for solving NP-complete absolute value equation, so that it is numerically stable and has convergence order two. Moreover, the convergence conditions are weaker than already iterative methods, hence this method can be applied to a broad range of problems. Applicability of the proposed method is tested for various examples.

1. Introduction

We consider the following absolute value equation (AVE)

$$G(x) = Ax - |x| - b = 0, \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$, and $|.|$ denotes absolute value. Mangasarian has proved that the general linear complementarity problem (LCP) is equivalent to an absolute value equation such as (1) (see Proposition in 1 [1]). To solve (1), Mangasarian applies the generalized Newton method for solving the AVE (1) provided that the singular values of A are not less than one (see Lemma 6 in [2]). Although, the generalized Newton method is a linear convergent method, a quadratically convergent method under the same condition has been developed [3]. When the singular values of A exceed 1, the AVE (1) has a unique solution [4,5]. It is worth pointing out that this condition has some equivalences [6]. Hence it seems that under this limitation, such iterative methods converge globally [7]. On the other hand, Prokopyev proves that checking whether the AVE (1) has a unique or multiple solutions is an NP-complete problem [8]. Therefore, it is not generally possible to construct a polynomial algorithm for solvability of AVE. It is worth noting that to avoid the assumption of having singular values greater than one, some other iterative methods have been developed in which all of them converge linearly [9–12].

We develop an iterative method to overcome the two limitations suggested by Mangasarian [1] of the generalized Newton method for solving the NP-complete AVE (1). First, since we are dealing with an NP-complete problem, we cannot generally assume that the singular values of A exceed one. As a vivid example in R, the generalized Newton–Mangasarian method fails to solve the AVE $x-|x| = 1$, because it has no solution. Consequently, as a limitation, this assumption can undoubtedly reduce the number of the real problems and applications that occur in LCP. Second, we focus on modifying the generalized method in such a way that has convergence order two, and it is a numerically stable method. Here our method converges locally because of the nonlinear and NP-complete nature of the problem. If we want to obtain a global quadratically convergent method, we need to make extra assumptions, or, we should consider very special cases. So, we wish to put it aside as an independent

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research problem. This paper is organized as follows: Section 2 deals with construction of the proposed method. Then the convergence analysis and numerical stability are presented. Section 3 is devoted to numerical test problems. The last section concludes the paper.

2. Main results

In this section, reconsidering the generalized Newton method [1], we modify it in such a way that it has convergence order two with some more general conditions compared with the given conditions by Managasarian in [1]. To this end, let the generalized Jacobian of (1) be given by [1]:

$$J_G(x) = A - T_z(x), \tag{2}$$

where $T_z(x)$ =diag(sign(x)). Let x^0 be a suitable starting vector to the exact solution, say x^*, of (1). Then, we propose the following modified Newton–Mangasarian method

$$(A - T_z(x^k))\Delta x^k = -Ax^k + |x^k| + b, \tag{3}$$

$$x^{k+1} = x^k + \Delta x^k, \quad k = 0, 1, 2, \ldots. \tag{4}$$

It should be noted that we first solve the linear system (3), and then, we update the value x^{k+1} from (4). Therefore, we reduce the numerical solution of solving a nonlinear system of equations to the numerical solution of a linear systems of equations. For more details, one can consult [13–15]. We will prove, under weaker conditions than given already, that this method is numerically stable and of convergence order two.

To prove the quadratic convergence order of the method (3)–(4), we need the following lemma:

Lemma 2.1. *Let D be an open convex set in R^n, and let J_G be Lipschitz continuous at x in the neighborhood D. Then, for any $t \in [0, 1]$ and $x + t\Delta x \in D$,*

$$\|G(x + \Delta x) - G(x) - J_G(x)\Delta x\| \le \frac{L_{J_G}}{2}\|\Delta x\|^2, \tag{5}$$

where L_{J_G} is the Lipschitz constant for J_G at x, in other words,

$$\|J_G(x + t\Delta x) - J_G(x)\| \le L_{J_G}\|t\Delta x\|.$$

Proof. By the use of integral mean value theorem, we have

$$\begin{aligned} G(x + t\Delta x) - G(x) - J_G(x)\Delta x &= \int_0^1 J_G(x + t\,\Delta x)\,\Delta x\,dt - J_G(x)\,\Delta x \\ &= \int_0^1 (J_G(x + t\,\Delta x) - J_G(x))\;\Delta x\,dt. \end{aligned}$$

Taking norm and considering the Lipschitz condition on J_G, we have

$$\begin{aligned} \|G(x + t\Delta x) - G(x) - J_G(x)\Delta x\| &= \|\int_0^1 \big(J_G(x + t\,\Delta x) - J_G(x)\big)\,\Delta x\,dt\| \\ &\le \int_0^1 \|J_G(x + t\,\Delta x) - J_G(x)\|\,\|\Delta x\|\,dt \\ &\le \int_0^1 \|L_{J_G}t\,\Delta x\|\,\|\Delta x\|\,dt = \frac{L_{J_G}}{2}\|\Delta x\|^2. \end{aligned}$$

Now, we can prove the quadratic convergence of the proposed method (3)–(4). Let $N_r(x^*) = \{x \in R^n : \|x - x^*\| < r\}$, and $r_k = \|x^k - x^*\|$.

Theorem 2.2. *Suppose that x^* is a solution of the AVE (1), i.e., $G(x^*) = 0$. In addition, suppose that the assumptions of the Lemma 2.1 hold, G is a continuously differentiable for all $x^k \in N_r(x^*) \subset D$, and $\|J_G(x)^{-1}\| < 1$. Then, the sequence $\{x^k\}$, $k > 0$, generated by (3)–(4) satisfies*

$$\|x^{k+1} - x^*\| \le \frac{L_{J_G}}{2}\|x^k - x^*\|^2. \tag{6}$$