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Boundedness of the density normalised Jones' square function does not imply 1-rectifiability

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ABSTRACT

Recently, M. Badger and R. Schul proved [4] that for a 1-rectifiable Radon measure μ, the density weighted Jones' square function

$$\widetilde{J}_2(x) = \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 1}} \beta_{2,\mu}^2(3Q) \frac{\ell(Q)}{\mu(Q)} 1_Q(x)$$

is finite for μ-a.e. x. Answering a question of Badger–Schul, we show that the converse is not true. Given $\epsilon > 0$, we construct a Radon probability measure on $[0,1]^2 \subset \mathbb{R}^2$ with the properties that $\widetilde{J}_2(x) \leq \epsilon$ for all $x \in \operatorname{spt} \mu$, but nevertheless the 1-dimensional lower density of μ vanishes almost everywhere. In particular, μ is purely 1-unrectifiable.

RÉSUMÉ

Récemment, M. Badger et R. Schul ont montré [4] que, si une mesure de Radon μ est 1-rectifiable, alors la fonctionnelle de Jones pondérée

$$\widetilde{J}_2(x) = \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 1}} \beta_{2,\mu}^2(3Q) \frac{\ell(Q)}{\mu(Q)} 1_Q(x)$$

est finie μ-p.p. Répondant à une question de Badger et Schul, nous montrons que la réciproque n'est pas vraie. Pour tout $\epsilon > 0$, nous construisons une mesure de Radon de probabilité sur $[0,1]^2 \subset \mathbb{R}^2$ telle que $\widetilde{J}_2(x) \leq \epsilon$ pour tout $x \in \operatorname{spt} \mu$, alors que néanmoins la densité inférieure unidimensionnelle de μ s'annule presque partout. En particulier, μ est purement 1-non rectifiable.

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1. Introduction

A Radon measure μ in $\mathbb{R}^2$ is 1-rectifiable if there exist countably many Lipschitz maps $f_i\colon \mathbb{R} \to \mathbb{R}^2$ such that

$$\mu\left(\mathbb{R}^2 \setminus \bigcup_i f_i(\mathbb{R})\right) = 0.$$

Recent years have seen lively interest in attempting to characterise the rectifiability of general Radon measures in terms of *β-numbers*, originally defined by P. Jones, G. David and S. Semmes. The existence of such a characterisation was conjectured by P. Jones around 2000. We start by mentioning a three-paper series of M. Badger and R. Schul [4–6], where the authors study the connection between 1-rectifiability and the boundedness of certain square functions, usually nicknamed *Jones' square functions*. A natural example of these objects is the following function $\widetilde{J}_2 := \widetilde{J}_{2,\mu}$, the *density normalised Jones' square function*:

$$\widetilde{J}_2(x) := \sum_{\substack{Q\in\mathcal{D}\\ \ell(Q)\le 1}} \beta^2_{2,\mu}(3Q)\frac{\ell(Q)}{\mu(Q)}1_Q(x), \quad x \in \mathbb{R}^2.$$

The notation $\widetilde{J}_2$ was introduced by Badger and Schul [4–6] – in contrast to the square function J_2 without the density normalisation factor $\ell(Q)/\mu(Q)$. Above, $\mathcal{D}$ is the standard dyadic grid in $\mathbb{R}^2$, μ is a Radon measure, and the β-numbers are defined as follows.

Definition 1.1 *(β-numbers).* Let μ be a Radon measure on $\mathbb{R}^2$. For a square $Q \subset \mathbb{R}^2$ with $\mu(Q) > 0$, we define the number $\beta_{2,\mu}(Q)$ by

$$\beta_{2,\mu}(Q) = \inf_L \left[\frac{1}{\mu(Q)}\int_Q \left(\frac{d(y,L)}{\mathrm{diam}(Q)}\right)^2 \,\mathrm{d}\mu(y)\right]^{1/2},$$

where the inf is taken over all (affine) lines $L \subset \mathbb{R}^2$.

This L^2 definition of β-numbers is due to G. David and S. Semmes [7,8]. The validation for $\widetilde{J}_2$ is, no doubt, the following theorem of Badger–Schul [4]: if μ is 1-rectifiable, then

$$\widetilde{J}_2(x) < \infty$$

for μ-a.e. x. This indicates that the pointwise μ-a.e. boundedness of $\widetilde{J}_2$ could, potentially, characterise the 1-rectifiability of a general Radon mesure μ, in the spirit of Jones' conjecture; in fact, Badger and Schul manage to prove this in [6] under the assumption that μ is pointwise doubling. In the present paper, we disprove the conjecture for general measures: the pointwise boundedness of $\widetilde{J}_2$ does not imply 1-rectifiability. This answers a question of Badger and Schul, see the discussion after Theorem E in [6]. In fact, the boundedness of $\widetilde{J}_2$ does not even imply that μ has non-vanishing 1-dimensional lower density

$$\Theta^1_*(\mu,x) := \liminf_{r\to 0}\frac{\mu(B(x,r))}{2r}$$

in a set of positive measure. Here is the precise statement:

Theorem 1.2. *Given $\epsilon > 0$, there exists a Radon probability measure μ supported on $[0,1]^2 \subset \mathbb{R}^2$ with the following properties:*