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Weak Milstein scheme without commutativity condition and its error bound

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ABSTRACT

This paper shows a discretization method of solution to stochastic differential equations as an extension of the Milstein scheme. With a simple method, we reconstruct weak Milstein scheme through second order polynomials of Brownian motions without assuming the Lie bracket commutativity condition on vector fields imposed in the classical Milstein scheme and show a sharp error bound for it. Numerical example illustrates the validity of the scheme.

1. Introduction

The Milstein scheme proposed in [17] is one of the well-known discretization scheme of stochastic differential equations (SDEs) as well as the Euler–Maruyama scheme [16]. The scheme is widely used in many application fields such as physics, biology, statistics and finance, see [1] [4] [7] [12] [19] for example. This paper focuses on a new fact on the Milstein scheme. To explain the feature of the Milstein scheme and motivation of this research, let us consider the solution $(X_t)_{t\geq 0}$ to the following Itô SDE

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{d} V_i(X_t)dW_t^i, \quad X_0 = x \in \mathbb{R}^N, \tag{1.1}$$

with d-dimensional Brownian motion $(W_t)_{t\geq 0} = (W_t^1, \cdots, W_t^d)_{t\geq 0}$ and appropriate smooth functions V_i, $i = 1, \cdots, d$ which are identified as vector fields on $\mathbb{R}^N$, i.e. $V_i\varphi(x) = \sum_{1\leq j\leq N} V_i^j(x)\frac{\partial}{\partial x_j}\varphi(x)$ for smooth function φ. Then X_t is approximated by stochastic Taylor expansion as

$$X_t \approx x + V_0(x)t + \sum_{j=1}^{d} V_j(x)W_t^j + \sum_{j_1,j_2=1}^{d} V_{j_1}V_{j_2}(x) \int_{0<t_1<t_2<t} dW_{t_1}^{j_1} dW_{t_2}^{j_2}. \tag{1.2}$$

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The expansion (1.2) consists basics of the Milstein scheme. The Milstein scheme is defined by

$$\overline{X}^{\mathrm{Mil},(n)}_{t_k} = \overline{X}^{\mathrm{Mil},(n)}_{t_{k-1}} + V_0(\overline{X}^{\mathrm{Mil},(n)}_{t_{k-1}})(t_k - t_{k-1}) + \sum_{j=1}^{d} V_j(X^{\mathrm{Mil},(n)}_{t_{k-1}})(W^j_{t_k} - W^j_{t_{k-1}})$$
$$+ \sum_{j_1,j_2=1}^{d} V_{j_1}V_{j_2}(\overline{X}^{\mathrm{Mil},(n)}_{t_{k-1}}) \int_{t_{k-1}}^{t_k} (W^{j_1}_s - W^{j_1}_{t_{k-1}})dW^{j_2}_s, \tag{1.3}$$

$0 = t_0 < t_1 < \cdots < t_n = T$, starting from $\overline{X}^{\mathrm{Mil},(n)}_{t_0} = x$. It is well known that the order of strong convergence of the scheme is $O(1/n)$ which is superior to that of the Euler–Maruyama scheme, and also it is known that the order of weak convergence is $O(1/n)$ but which is same to the Euler–Maruyama scheme, see [12] for instance. Then, it is commonly recognized that the Milstein scheme does not improve the rate of convergence of the Euler–Maruyama scheme in weak approximation sense.

We recall how the Milstein scheme is simulated. The Milstein scheme is implemented using Brownian motions if a limited condition is satisfied. The main difficulty in the implementation comes from the term $\int_{t_{k-1}}^{t_k}(W^{j_1}_s - W^{j_1}_{t_{k-1}})dW^{j_2}_s$, $1 \le j_1, j_2 \le d$. There is no exact density of this iterated integral when $d \ge 2$ and $j_1 \ne j_2$. For example, see [7] [12] [24] for the details on this problem. However, when the vector fields satisfy the *commutativity condition*, we can avoid direct simulation of the term $\int_{t_{k-1}}^{t_k}(W^{j_1}_s - W^{j_1}_{t_{k-1}})dW^{j_2}_s$, $1 \le j_1, j_2 \le d$. Let $[A, B](x) = \sum_{i=1}^N A^i(x)\frac{\partial}{\partial x_i}B(x) - B^i(x)\frac{\partial}{\partial x_i}A(x)$ be the Lie bracket of smooth functions $A, B : \mathbb{R}^N \to \mathbb{R}^N$, and consider the commutativity condition

$$[\mathbf{C}] \qquad [V_i, V_j](x) = 0, \ \text{ for all } \ i, j = 1, \cdots, d \ \text{ and } x \in \mathbb{R}^N.$$

Then under **[C]**, the scheme (1.3) is simplified as

$$\overline{X}^{\mathrm{Mil},(n)}_{t_k} = \overline{X}^{\mathrm{Mil},(n)}_{t_{k-1}} + V_0(\overline{X}^{\mathrm{Mil},(n)}_{t_{k-1}})(t_k - t_{k-1}) + \sum_{j=1}^{d} V_j(X^{\mathrm{Mil},(n)}_{t_{k-1}})(W^j_{t_k} - W^j_{t_{k-1}})$$
$$+ \sum_{j_1,j_2=1}^{d} V_{j_1}V_{j_2}(\overline{X}^{\mathrm{Mil},(n)}_{t_{k-1}}) \frac{1}{2}\{(W^{j_1}_{t_k} - W^{j_1}_{t_{k-1}})(W^{j_2}_{t_k} - W^{j_2}_{t_{k-1}}) - (t_k - t_{k-1})\mathbf{1}_{j_1=j_2}\}. \tag{1.4}$$

The construction of the simpler Milstein scheme (1.4) is mainly due to the following relation through Itô's product formula: for $i, j = 1, \cdots, d$,

$$W^i_t W^j_t = \int_0^t \int_0^s dW^i_u dW^j_s + \int_0^t \int_0^s dW^j_u dW^i_s + t\mathbf{1}_{i=j}. \tag{1.5}$$

Then the commutativity condition **[C]** is crucial because this enables us to replace iterated integrals in (1.2) with second order polynomials of Brownian motions using (1.5) and thereby the simpler Milstein scheme holds. The univariate SDE $N = d = 1$ always satisfies the commutativity condition **[C]**. In general, when **[C]** is not satisfied, the scheme (1.4) does not make sense since we can not use the property of Brownian motions (1.5) anymore. Also, even if the condition **[C]** holds, the order of the Milstein scheme is same as that of the Euler–Maruyama scheme in weak sense, as mentioned above. We refer to further development on the Milstein scheme as [2] [3].

Although these are common understanding on the Milstein scheme, we would see some new facts under suitable conditions:

1. the Milstein scheme (1.4) still holds using weak approximation analysis even if the commutativity condition **[C]** is not satisfied,
2. the approximation error of the simpler Milstein scheme without commutativity condition would be sharper than that of the Euler–Maruyama scheme, while the order of discretization itself is same.

We prove these conjectures using a simple technique from Malliavin calculus.

The organization of the paper is as follows. In Section 2, we show extension of the Milstein scheme as weak approximation after we prepare some key results on small time approximations. Section 3 shows numerical examples in finance to confirm the validity and the effectiveness of the proposed scheme. The proofs of key statements are left for Appendix.

2. Extension of Milstein scheme

Let $C^k_b(\mathbb{R}^N)$ be the space of k-times continuously differentiable functions $f : \mathbb{R}^N \to \mathbb{R}$ such that f and its derivatives are bounded and $C_{Lip}(\mathbb{R}^N)$ be the space of Lipschitz continuous functions on $\mathbb{R}^N$ into $\mathbb{R}$. We denote by $\|f\|_\infty = \sup_{x\in\mathbb{R}^N} |f(x)|$