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# A finite element method for a seawater intrusion problem in unconfined aquifers

A. Abudawia [a], A. Mourad [a,b,d], J.H. Rodrigues [c], C. Rosier [a,b,*]

[a] ULCO, LMPA J. Liouville, BP 699, F-62 228 Calais, France
[b] CNRS FR 2956, France
[c] State University of Londrina, CEP 86057-970, Londrina, Paraná, Brazil
[d] LaMA-Liban, Lebanese University, P.O. Box 37 Tripoli, Lebanon

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ABSTRACT

We simulate a sharp-diffuse interface model issuing from a seawater intrusion problem in unconfined aquifer. We study a semi-implicite in time scheme for a $P_k$, $(k \geq 1)$ Lagrange finite element approximation. Using the specific regularity of the exact solution, we state that the scheme is of order 1 in time and $k$ in space. We propose a finite volume method for a regular mesh and we compare the results given by these two approximations.



## 1. Introduction

In this paper, we are interested in the optimal exploitation of fresh water in coastal zones. Indeed coastal zones are densely populated areas and the intensive extraction of fresh water yields to a local water table depression causing sea intrusion problems. In order to control these seawater intrusions, we need efficient and accurate models to simulate the transport of salt water front in coastal aquifers. We consider unconfined aquifers which are bounded by two layers: the lower layer is always supposed to be impermeable and the upper surface is a permeable layer constituted by gravels, sand or alluvia. The basis of the modeling is the mass conservation law written for each species (fresh and salt water) combined with the classical Darcy law for porous media. In the present work we have essentially chosen to adopt a sharp interface approach, based on the assumption that the two fluids are immiscible. We assume that each fluid is confined to a well defined portion of the flow domain with a smooth interface separating them called sharp interface, effects of capillary pressure type are thus neglected. This approximation is often reasonable (see e.g. [5] and below, [9,20,27]). Following [12], we can mix this abrupt interface approach with a phase field approach (here an Allen–Cahn type model in fluid–fluid context, see e.g. [2,3,8,15]) in re-including the existence of a diffuse interface between fresh and salt water where mass exchanges occur. In such a way, we combine the advantage of respecting the physics of the problem and that of the computational efficiency. The same process is also applied to model the transition between the saturated and unsaturated zones.

The evolutions of the depths of the free interfaces $h$ and $h_1$ are given by a $2D$ strongly coupled system of nonlinear parabolic equations. From a theoretical point of view, two advantages resulting from the addition of diffuse areas compared

* Corresponding author.
*E-mail addresses:* abudawia.amel@yahoo.com (A. Abudawia), mourad@univ-littoral.fr (A. Mourad), jh.rodrigues@ymail.com (J.H. Rodrigues), rosier@univ-littoral.fr (C. Rosier).

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to the sharp interface approximation are stated in [10]: if diffuse interfaces are both present, the system has a parabolic structure, the degeneracy appearing only in the sharp interface case. This allows us to demonstrate a more efficient and logical maximum principle from the physical point of view (see for instance [17,26]). But above all, we can prove that the solution belongs to the space $L^r(0,T;W^{1,r}(\Omega))$ for some $r>2$. This regularity allows us to manage the nonlinearity of the system, the main consequence being then the uniqueness of the solution. It should be noted that the assumption ensuring the uniqueness result is very restrictive from the physical point of view, namely we assume a very low hydraulic conductivity inside the aquifer (cf. [13]). In this article, we are only interested in the model resulting from a sharp-diffuse interface approach in unconfined aquifers. More precisely, if $\delta$ denotes the thickness of the diffuse interfaces, assumed positive throughout, this necessary parameter guarantees the regularity of the solution and allows the convergence analysis of the finite element scheme presented here.

In the first part of the paper, we study a semi-implicit in time scheme combined with a $P_k\,(k\geq 1)$ finite element method to discretize our problem. A first study concerning a finite element method applied to seawater intrusion in confined aquifer is done in [1]. In that case, the model consists in a coupled system of nonlinear elliptic–parabolic equations describing the evolutions of $h$ and of the freshwater hydraulic head $\phi_f$. Assuming $(h,\phi_f)\in C([0,T],H^3(\Omega))\big)^2$, it was established in [1] that the $P_1$ Lagrange finite element scheme is of order 1 in time and in space under restrictive assumptions on physical parameters. Moreover, it was considered a classical experiment where the freshwater/saltwater interface is described by a linear profile pivoting around a fixed point (see [17,24]). Thanks to the analytic solution proposed by Keulegan, numerical comparisons were performed, thus illustrating the convergence order of the finite element method.

In this paper, we improve the convergence result given in [1] since the present proof of the convergence result is based on the specific properties of the exact solution instead of supplementary regularity assumptions. Namely, we use the maximum principle satisfied by the solution and the bound in $L^r$-norm $(r>2)$ of the gradient of the solution. Thanks to the Gagliardo–Nirenberg inequality written for space $L^4$, we can handle the nonlinearity of the problem. We emphasize that the regularity required for $(h,h_1)$ in the error estimates results is the minimal regularity necessary to establish interpolation error estimates, i.e. $(h,h_1)\in C^1([0,T],H^{k+1}(\Omega))$, $k\geq 1$ ([16], Thm. 2.7.4, Chapter 2). Nevertheless, in each case (confined and unconfined), we need the same kind of stability condition between the time step and the space discretization to ensure the convergence of the scheme. It should be noted that the theoretical stability condition assumed in our convergence theorem is very constraining. But in practice, it is enough to take a time step of the order of the square of the space step.

Contrary to the confined case, we do not have a test case by which we could illustrate totally the convergence order proved for the finite element method given in this paper. More precisely, if we impose a Dirichlet boundary condition on the left boundary to the saltwater elevation h, we can compare the numerical approximation of $h$ to the solution computed with the Ferris model. But, this comparison is not possible for $h_1$, since there is no analytic solution. On the other hand, we know that if the hydraulic conductivity is sufficiently low, the exact problem admits a unique solution. Given this result, the convergence theorem makes it possible to conclude that the numerical solution is a suitable approximation of the unique exact solution, but only if the hydraulic conductivity is assumed to be low. To confirm the numerical results in more realistic conditions from a physical point of view, we propose to compare them with those obtained by another numerical approximation. To this end, we present a finite volume scheme based on the intrinsic nature of the model that derives from conservative laws. The main aim is to validate the numerical results obtained with the finite element scheme. The numerical simulations show that the two numerical methods lead to similar qualitative results when the system evolves without forcing term. In the cases of injection or pumping scenario, we need to refine the space discretization in order to get a relative difference between the two numerical solutions in $L^\infty$ norm of order $10^{-2}$.

The outline of the paper is the following one. Section 2 is devoted to the model and its derivation. In section 3 we recall all mathematical notations, the global in time existence result and the regularity result verified by the solution. In section 4 we describe the $P_1$ Lagrange finite element scheme and we prove the existence and uniqueness of the numerical solution. We then state the main result establishing that the scheme is of order 1 in time and $k$ in space. Section 5 is devoted to the presentation of the finite volume method with a regular mesh. In section 6, we perform numerical simulations illustrating the convergence analysis presented in section 4. First, we impose, on the left of the domain, oscillating Dirichlet boundary conditions in order to simulate the tidal effects on the aquifer. Given the analytic solution deduced from Ferris model, we compute the error in $L^\infty$ norm and in $L^2$ norm for the saltwater elevation. Then, we perform numerical experiments in order to compare the approximate solutions obtained with the finite element scheme and the finite volume scheme. We next consider homogeneous Neumann boundary conditions, in such a way that the system can freely evolve. Three scenarios are considered: the situation without forcing term, another with an injection of freshwater in a part of the aquifer, and the last one corresponding to the freshwater pumping.

## 2. Modeling

Introducing specific index for the fresh $(f)$ and salt $(s)$ waters, we write the mass conservation law for each species (fresh and salt water) combined with the classical Darcy law for porous media. Hydraulic heads $\Phi_i$, $i=f,s$ are defined at elevation $z$ by

$$\Phi_i=\frac{P_i}{\rho_i g}+z,$$