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Automorphism groups of graph covers and uniform subset graphs

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Abstract

Hofmeister considered the automorphism groups of antipodal graphs through the exploration of graph covers. In this note we extend the exploration of automorphism groups of distance preserving graph covers. We apply the technique of graph covers to determine the automorphism groups of uniform subset graphs $\Gamma(2k, k, k-1)$ and $\Gamma(2k, k, 1)$. The determination of automorphism groups answers a conjecture posed by Mark Ramras and Elizabeth Donovan. They conjectured that $\mathrm{Aut}(\Gamma(2k, k, k-1)) \cong S_{2k} \times < T >$, where T is the complementation map $X \mapsto T(X) = X^c = \{1, 2, \ldots, 2k\} \setminus X$, and X is a k-subset of $\Omega = \{1, 2, \ldots, 2k\}$.

Keywords: Graph covers; Antipodal graphs; Uniform subset graphs

1. Introduction

In this paper, all graphs considered are assumed to be finite and simple. For a graph Γ, we let $V(\Gamma)$, $E(\Gamma)$, $\mathrm{Aut}(\Gamma)$ and $e = \{x, y\}$ denote the vertex set, the edge set, the automorphism group and an edge with endpoints x and y, respectively. A graph Γ is **vertex-transitive** if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. For a graph Γ, $x \in V(\Gamma)$, we set $\Gamma_i(x) := \{y \in V(\Gamma) | d(x, y) = i\}$ and $\epsilon(x) = \max\{d(x, y) | y \in V(\Gamma)\}$, where d is the usual shortest distance path in Γ. Let $\mathcal{P}$ be a partition of $V(\Gamma)$. By the quotient graph $\Gamma/\mathcal{P}$ is meant the graph with

$$V(\Gamma/\mathcal{P}) := \mathcal{P};$$

$$\{X, Y\} \in E(\Gamma/\mathcal{P}) \iff X \neq Y \text{ and } \{x, y\} \in E(\Gamma) \text{ for some } x \in X, y \in Y.$$

A graph Γ is said to be **antipodal** if the collection of sets $\{x\} \cup \Gamma_{\epsilon(x)}(x)$ is a partition of $V(\Gamma)$.

For a graph Γ and $A \subset V(\Gamma)$, the minimum distance of A is defined by $d(A) := \min_{x,y,x \neq y \in A} d(x, y)$.

Chen and Lih [1] introduced uniform subset graphs. This is a generalisation of such graphs as Kneser graphs and Johnson graphs. They are defined by the following. Let n, k be positive integers such that $n \geq 2k$, and i a non-negative

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integer with $i < k$. Let $\Omega = \{1, 2, \ldots, n\}$ and $\Omega^{\{k\}}$ the set of all k-subsets of Ω. A **uniform subset graph** $\Gamma(n, k, i)$ is defined by

$$V(\Gamma(n, k, i)) = \Omega^{\{k\}};$$

$$\{X, Y\} \in E(\Gamma(n, k, i)) \iff |X \cap Y| = i, \ X, Y \in \Omega^{\{k\}}.$$

Let Ω be a non-empty set. S_Ω denotes the set of permutations of the set Ω. If $\Omega = \{1, 2, \ldots, n\}$, S_Ω is written S_n. Since S_n is k-transitive for all $k \leq n$ and preserves the size of intersection sets, it is easy to see that uniform subset graphs are vertex transitive. However, it is surprisingly difficult to determine their full automorphism groups. Amongst the many classes of uniform subset graphs it has only been determined that $\mathrm{Aut}(\Gamma(2k+1, k, 0))$, the automorphism group of the so called Odd graphs, is S_n, and more recently, Ramras and Donovan [2] proved that $\mathrm{Aut}(\Gamma(n, k, k-1))$, $n \neq 2k$ coincides with S_n. Further, they conjectured that $\mathrm{Aut}(\Gamma(2k, k, k-1)) \cong S_n \times \langle T \rangle$, where T is the complementation map $X \mapsto T(X) = X^c = \{1, 2, \ldots, n\} \setminus X$, and $X \in \Omega^{\{k\}}$.

In this note we determine the automorphism groups of the uniform subset graphs $\Gamma(2k, k, 1)$ and $\Gamma(2k, k, k-1)$.

2. Automorphisms of graph covers

In order to determine the automorphism groups of the graphs in question, we employ Hofmeister's [3] strategy. He determines the automorphism group of a graph cover by first looking at the quotient (folded) graph. The key observation in analysing the automorphism group of the cover is in understanding the interplay between automorphisms of the cover and their corresponding quotient.

Definition 1. Let Γ and Δ be graphs and $r \geq 2$ be an integer. Δ is called an r-cover of Γ if there is an epimorphism $\rho : \Delta \to \Gamma$, called the covering projection, such that

(i) $|\rho^{-1}(x)| = r$, for every $x \in V(\Gamma)$;
(ii) ρ bijectively sends $\Delta_1(x)$ to $\Gamma_1(\rho(x))$ for each $x \in V(\Gamma)$.

The graph Γ is called the **fold** of Δ.

Gross and Tucker [4] have shown that graph covers arise from permutation voltage graphs so that the consideration of the former amounts to focusing on the later. Permutation voltage graphs are defined in the following.

For a graph Γ, let $A(\Gamma)$ be the arc set of the corresponding symmetric digraph. A permutation voltage assignment in a symmetric group S_r for Γ is a mapping $f : A(\Gamma) \to S_r$ such that $f((x, y)) = (f((y, x)))^{-1}$, for any arc (x, y) in $A(\Gamma)$.

Given a graph Γ and a permutation voltage assignment f, the derived graph Γ_f is the graph with

$$V(\Gamma_f) = V(\Gamma) \times \{1, 2, \ldots, r\};$$

$$\{(x, i), (y, j)\} \in E(\Gamma_f) \iff \{x, y\} \in E(\Gamma), (f(x, y))i = j.$$

Gross and Tucker's reductions, as alluded to, is the content of the following two results.

Theorem 1 ([4]). *Let Γ be a graph and $f : A(\Gamma) \to S_r$ a permutation voltage assignment. Then the natural projection $\rho_f : \Gamma_f \to \Gamma$ (sending vertex (u, i) of Γ_f to vertex u of Γ) is an r-fold covering projection.*

Theorem 2 ([4]). *Let $\rho : \Delta \to \Gamma$ be a r-fold covering projection. Then there is an assignment f of voltages in the symmetric group S_r for Γ such that the covering projections ρ and ρ_f are isomorphic with respect to the trivial automorphism.*

It is in the context of permutation voltage graphs that Hofmeister [3] considered the interplay between the automorphisms of graph covers and their corresponding quotient graphs. The essence of this interplay is in the following. Let $H \leq \mathrm{Aut}(\Gamma)$ be a group of automorphisms of the graph Γ. A H-automorphism of a covering projection $\rho : \Delta \to \Gamma$ is a pair (φ, ψ), consisting of an automorphism $\varphi \in H$ and an automorphism $\psi : \Delta \to \Delta$, such that $\varphi\rho = \rho\psi$.

As a generalisation, we consider r-coverings that are defined by semi-regular automorphisms in vertex-transitive graphs. A semi-regular element of an automorphism group of a graph is a non-identity element having all cycles of