ARTICLE IN PRESS

Available online at www.sciencedirect.com
ScienceDirect

AKCE International Journal of Graphs and Combinatorics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

AKCE International Journal of Graphs and Combinatorics

www.elsevier.com/locate/akcej

Edges and vertices in a unique signed circle in a signed graph

Richard Behr

Department of Mathematics, Binghamton University, Binghamton, NY 13902, United States

Received 31 December 2016; accepted 6 March 2017
Available online xxxxx

Abstract

We examine the conditions under which a signed graph contains an edge or a vertex that is contained in a unique negative circle or a unique positive circle. For an edge in a unique signed circle, the positive and negative case require the same structure on the underlying graph, but the requirements on the signature are different. We characterize the structure of the underlying graph necessary to support such an edge in terms of bridges of a circle. We then use the results from the edge version of the problem to help solve the vertex version.

Keywords: Signed graph; Balance; Negative circle; Positive circle; Bridge

0. Introduction

A *signed graph* is a graph in which each edge is assigned either a positive or negative sign. The sign of a circle (a connected, 2-regular subgraph) in a signed graph is defined to be the product of the signs of its edges. In many cases, the most important feature of a signed graph is the sign of each of its circles. A signed graph that contains no negative circle is said to be *balanced*, while a signed graph that contains at least one negative circle is *unbalanced*. The purpose of this paper is to determine when a signed graph contains an edge or a vertex that is contained in a unique negative circle or a unique positive circle.

Signed graphs were invented by Harary in 1953 in order to help study a question in social psychology [1]. In 1956, Harary observed that an edge of a signed graph lies in some negative circle if and only if the block (maximal 2-connected subgraph) containing it is unbalanced [2]. Similarly, an edge lies in some positive circle if and only if it is not a balancing edge in its block (see Lemma 3.3). Our problem is related to these facts, but the added uniqueness condition creates many additional restrictions on both the structure of the underlying graph and the signature.

Peer review under responsibility of Kalasalingam University.
E-mail address: behr@math.binghamton.edu.

http://dx.doi.org/10.1016/j.akcej.2017.03.004

1. Definitions

1.1. Graphs

A *graph* $G = (V(G), E(G))$ consists of a finite vertex set $V(G)$ and finite edge set $E(G)$. Each edge has a pair of vertices as its *endpoints*, and we write $e{:}uv$ for an edge with endpoints u and v. A *link* is an edge with two distinct endpoints, and a *loop* has two equal endpoints. We write K_n for the complete graph on n vertices.

Let H be a subgraph of G. Then for $v \in V(G)$, the *degree of* v *in* H, denoted as $\deg_H(v)$, is the number of edges in H that are incident with v (a loop counts twice).

A *circle* C is a connected 2-regular subgraph. An edge $e \in E(G) \backslash E(C)$ connecting two different vertices of C is a *chord*.

A *path* $P = v_0, e_0, v_1, e_1, \ldots, e_{n-1}, v_n$ is a sequence of adjacent vertices and connecting edges that never repeats an edge or a vertex. We call v_0 and v_n the *endpoints* of P, while the other vertices are *interior vertices*.

We *subdivide* an edge by replacing it with a path that has at least one edge. A *subdivision* of G is a graph obtained by subdividing some of the edges of G.

Given a circle C of G, a *bridge* of C is either a connected component D of $G \backslash V(C)$ along with all edges joining D to C, or a chord of C. The *vertices of attachment* of a bridge D are the vertices in $V(D) \cap V(C)$. A path contained in D that has different vertices of attachment for its endpoints is a *path through* D.

A *cutpoint* of G is a vertex v with the property that there exist subgraphs H_1 and H_2 each with at least one edge, such that $G = H_1 \cup H_2$ and $H_1 \cap H_2 = \{v\}$. An *isthmus* is an edge whose removal increases the number of connected components. A *block* of G is a maximal subgraph that contains no cutpoint. Each edge is contained in exactly one block.

1.2. Signed graphs

A *signed graph* Σ is a pair (G, σ), where G is a graph (called the *underlying graph*), and $\sigma : V(G) \to \{+, -\}$ is the *signature*.

The *sign* of a circle C in Σ is defined to be the product of the signs of its edges. Thus, a signed circle can be either positive or negative. A signed graph is *balanced* if all of its circles are positive, and *unbalanced* if it contains at least one negative circle.

A *theta graph* consists of three paths with the same endpoints and no other vertices in common. The most useful thing about theta graphs in our context is the *theta property*: every signed theta graph has either 1 or 3 positive circles. If two circles C_1 and C_2 intersect in a path with at least one edge, then $C_1 \cup C_2$ is a theta graph with third circle $C_1 \Delta C_2$ (we use Δ for symmetric difference). By the theta property, if C_1 and C_2 have the same sign then $C_1 \Delta C_2$ is positive, and otherwise $C_1 \Delta C_2$ is negative.

A *switching function* on $\Sigma = (G, \sigma)$ is a function $\zeta : V(G) \to \{+, -\}$. We can use ζ to modify σ, obtaining a new signature given by $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where v, w are the endpoints of e. The switched signed graph is written $\Sigma^\zeta := (G, \sigma^\zeta)$. If Σ' is obtained from Σ via switching, we say Σ' and Σ are *switching equivalent*, written $\Sigma' \sim \Sigma$. Switching is useful for us because of the following fact.

Lemma 1.1 (*Zaslavsky [3], Sozański [4]*)**.** *Let Σ_1 and Σ_2 be signed graphs on the same underlying graph. Then, $\Sigma_1 \sim \Sigma_2$ if and only if Σ_1 and Σ_2 have the same collection of positive circles. In particular, Σ is balanced if and only if it switches to an all-positive signature.*

If Σ can be switched so that it has a single negative edge b, we call b a *balancing edge*. The deletion of a balancing edge yields a balanced signed graph. Moreover, if b is a balancing edge, the negative circles of Σ are precisely those that contain b.

Assume an edge e is contained in at least one circle. Then e is contained in only positive circles if and only if the block containing it is balanced, and e is contained in only negative circles if and only if it is a balancing edge in its block (Lemma 3.3). In other words, e is contained in some negative circle if and only if the block containing e is unbalanced (as discovered by Harary [2]), and e is contained in some positive circle if and only if e is not balancing in the block containing it.