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Which cospectral graphs have same degree sequences[☆]

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ABSTRACT

Two graphs are said to be L-cospectral (respectively, Q-cospectral) if they have the same (respectively, signless) Laplacian spectra, and a graph G is said to be $L-DS$ (respectively, $Q-DS$) if there does not exist other non-isomorphic graph H such that H and G are L-cospectral (respectively, Q-cospectral). Let $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$ be the degree sequence of a graph G with n vertices. In this paper, we prove that except for two exceptions (respectively, the graphs with $d_1(G) \in \{4, 5\}$), if H is L-cospectral (respectively, Q-cospectral) with a connected graph G and $d_2(G) = 2$, then H has the same degree sequence as G. A spider graph is a unicyclic graph obtained by attaching some paths to a common vertex of the cycle. As an application of our result, we show that every spider graph and its complement graph are both $L-DS$, which extends the corresponding results of Haemers et al. (2008), Liu et al. (2011), Zhang et al. (2009) and Yu et al. (2014).

1. Introduction

Throughout this paper, G is an undirected simple graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $d_G(v)$ and $N_G(v)$ be the degree and neighbor set of vertex v in G, respectively. If $d_G(v) = 1$, then we call v a *pendent vertex*. A vertex with maximum degree of G is always referred to be a *maximum vertex*. In the sequel, we enumerate the degrees of G in non-increasing order, i.e., $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$.

As usual, $K_{1,n-1}$, K_n, C_n and P_n denote the star, complete graph, cycle and path with n vertices, respectively. A connected graph G with n vertices and $n + c - 1$ edges is called a *c-cyclic graph*. An *s-rose graph* is a graph with $s\,(\geq 2)$ cycles that all meet in one vertex. Clearly, an s-rose graph is connected and the maximum vertex has degree $2s$ and the other vertices have degree 2. Let $R(p, q)$ be a 2-rose graph with two cycles C_p and C_q.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal degree matrix of G, respectively. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. It is easy to see that $Q(G)$ is positive semidefinite (see [18]) and hence its eigenvalues can be arranged as:

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) \geq 0.$$

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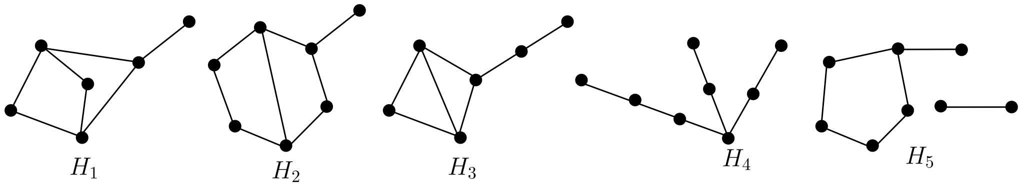

Fig. 1.1. The graphs H_1, H_2, H_3, H_4 and H_5.

Note that $L(G)$ is a positive semidefinite matrix with all row sums being equal to zero. Thus, we can use

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0$$

to denote the eigenvalues of $L(G)$. For the sake of brevity, if there is no risk of confusion, we always simplify $d_i(G)$, $d_G(v)$, $N_G(v)$, $\lambda_i(G)$ and $\mu_i(G)$ as d_i, $d(v)$, $N(v)$, λ_i and μ_i, respectively.

Two graphs are said to be *A-cospectral* (respectively, *Q-cospectral*, *L-cospectral*) if they have the same adjacency (respectively, signless Laplacian, Laplacian) spectra. Similarly, two graphs are said to be *co-degree* if they have the same degree sequences. A graph G is said to be $A - DS$ (respectively, $Q - DS$, $L - DS$) if there does not exist other non-isomorphic graph H such that H and G are A-cospectral (respectively, Q-cospectral, L-cospectral).

Which graphs are determined by their spectra? This question was proposed by Günthard and Primas [4] more than 50 years ago in the context of Hückel's theory in chemistry, and had drawn more and more attention recently. For details, we refer the readers to [2,16] and the references therein. In this line, it is an interesting and challenging problem that

Problem 1.1 (*[2]*)**.** How much is the ratio between these *DS* graphs and all graphs with n vertices when n tends to infinity?

Furthermore, as discovered in [23], when we want to prove that G is $Q - DS$ (respectively, $L - DS$), we always first prove that H and G are co-degree under the supposition of G and H being Q-cospectral (respectively, L-cospectral), and then prove that G and H are isomorphic. But the first step is not an easy task, and we always have to spend a large amount of energy (sometimes with the similar method) to solve it. Thus, one of the present authors and his cooperators put forward the following problem:

Problem 1.2 (*[23]*)**.** Which graphs satisfy the property "every Q-cospectral graph of G is also co-degree with G"?

Up to now, few results have been determined for Problem 1.1 [2], and it is far to be solved. Motivated by Problems 1.1 and 1.2, the following two problems are also interesting and they are much easier than Problem 1.1:

Problem 1.3. How much is the ratio between these graphs satisfying Problem 1.2 and all graphs with n vertices when n tends to infinity?

Problem 1.4. How much is the ratio between these degree sequences whose corresponding graphs have the same (respectively, signless) Laplacian spectra and all the graphic degree sequences for graphs with n vertices when n tends to infinity?

It is natural for us to consider the similar problems for adjacency spectra. There is exactly one connected graph with n vertices and $d_2 = 1$, that is $K_{1,n-1}$. It has been proved that $K_{1,n-1}$ is $L - DS$ (respectively, $Q - DS$ for $n \neq 4$) [14]. However, there are infinite connected graphs with n vertices and $d_2 = 2$ when n tends to infinity, and we shall prove the following two results in the sequel.

Theorem 1.1. *Let G be a connected graph with $d_2 = 2$ and $G \notin \{R(3,4), R(3,5)\}$. If G and H are L-cospectral, then H is connected and H is co-degree with G.*

Theorem 1.2. *Let G be a connected graph with $d_2 = 2$ and $d_1(G) \notin \{4, 5\}$. If G and H are Q-cospectral, then H is co-degree with G.*

Remark 1.1. Let $H_1, H_2, \ldots, H_5$ be the graphs as shown in Fig. 1.1. It has been shown that $R(3,4)$ and H_1 are L-cospectral [8], $R(3,5)$ and H_2 are L-cospectral [8], $R(3,4)$ and H_3 are Q-cospectral [12]. Furthermore, there are infinite pairs of non-isomorphic graphs such that they are both Q-cospectral and co-degree [19]. For instance, H_4 and H_5 are such a pair of graphs.

Let G be a graph such that every Q-cospectral (respectively, L-cospectral) graph of G is also co-degree with G. If the degree sequence of G is π and the graph with π as its degree sequence is unique, then G is $Q - DS$ (respectively, $L - DS$). For instance, by Theorems 1.1 and 1.2, we can easily prove that: if G is a connected graph with n vertices, $d_1 = n - 1$ and $d_2 = 2$, then G is $Q - DS$ and $L - DS$. That is the main result of [14].