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Partitioning edge-coloured complete symmetric digraphs into monochromatic complete subgraphs

Carl Bürger [a], Louis DeBiasio [b], Hannah Guggiari [c,*], Max Pitz [a]

[a] University of Hamburg, Department of Mathematics, Bundesstraße 55 (Geomatikum), 20146 Hamburg, Germany
[b] Miami University, Department of Mathematics, Oxford, OH, 45056, United States
[c] University of Oxford, Mathematical Institute, Oxford, OX2 6GG, United Kingdom

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ABSTRACT

Let $\vec{K}_{\mathbb{N}}$ be the complete symmetric digraph on the positive integers. Answering a question of DeBiasio and McKenney, we construct a 2-colouring of the edges of $\vec{K}_{\mathbb{N}}$ in which every monochromatic path has density 0.

However, if we restrict the length of monochromatic paths in one colour, then no example as above can exist: We show that every $(r+1)$-edge-coloured complete symmetric digraph (of arbitrary infinite cardinality) containing no directed paths of edge-length ℓ_i for any colour $i \leq r$ can be covered by $\prod_{i\in[r]}\ell_i$ pairwise disjoint monochromatic complete symmetric digraphs in colour $r+1$.

Furthermore, we present a stability version for the countable case of the latter result: We prove that the edge-colouring is uniquely determined on a large subgraph, as soon as the upper density of monochromatic paths in colour $r+1$ is bounded by $\prod_{i\in[r]}\frac{1}{\ell_i}$.

1. Introduction

Let $K_{\mathbb{N}}$ be the complete graph on the positive integers and $\vec{K}_{\mathbb{N}}$ be the complete symmetric digraph on the positive integers. The *upper density* of a set $A \subseteq \mathbb{N}$ is

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}$$

For a graph or digraph G with vertex set $V(G) \subseteq \mathbb{N}$, we define the *upper density* of G to be that of $V(G)$. We will write $[k]$ for the set $\{1, \dots, k\}$. Throughout this paper, by a k-colouring, we mean a k-edge-colouring. In a 2-colouring, we will assume that the colours are red and blue. Given a directed graph D and sets $A, B \subseteq V(D)$, we write $[A, B]$-edges to mean all edges $(x, y) \in E(D)$ with $x \in A$ and $y \in B$.

For finite graphs, Gerencsér and Gyárfás [7] proved that in every 2-colouring of K_n there is a monochromatic path on at least $(2n + 1)/3$ vertices; furthermore, this is best possible. Erdős and Galvin [5] proved an infinite analogue, showing that in every 2-colouring of $K_{\mathbb{N}}$ there exists a monochromatic infinite path with upper density at least $\frac{2}{3}$; they also give a 2-colouring of $K_{\mathbb{N}}$ in which every monochromatic path has upper density at most $\frac{8}{9}$. Recently, DeBiasio and McKenney [3] improved the lower bound, showing that, in every 2-colouring of $K_{\mathbb{N}}$, there exists a monochromatic infinite path with upper density at least $\frac{3}{4}$. For more colours, Rado [10] showed that, in every r-colouring of $K_{\mathbb{N}}$, there is a partition of the vertices into

* Corresponding author.
E-mail addresses: carl.buerger@uni-hamburg.de (C. Bürger), debiasld@miamioh.edu (L. DeBiasio), guggiari@maths.ox.ac.uk (H. Guggiari), max.pitz@uni-hamburg.de (M. Pitz).

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at most r disjoint paths of distinct colours,[1] which implies that there is a monochromatic path of upper density at least $\frac{1}{r}$. Elekes, Soukup, Soukup and Szentmiklòssy [4] have recently extended Rado's result for two colours to the complete graph on $\aleph_1$ where $\aleph_1$ is the smallest uncountable cardinal. Shortly after, Soukup [13] extended this further to any finite number of colours and complete graphs of any infinite cardinality.

For directed graphs, the picture is a little different. In the finite case, Raynaud [11] showed that, in any 2-colouring of $\vec{K}_n$, there is a directed Hamiltonian cycle C with the following property: there are two vertices a and b such that the directed path from a to b along C is red and the directed path from b to a along C is blue. As a corollary, in any 2-colouring of $\vec{K}_n$, there is a monochromatic directed path on at least $\frac{n}{2}+1$ vertices; this is easily seen to be best possible by partitioning the vertices of $\vec{K}_n$ into two sets A, B with $||A|-|B|| \leq 1$ and colouring the edge (x, y) red if $x \in A$ and blue if $x \in B$. In this paper, we will be interested in the infinite directed case. In particular, we will be considering edge-colourings of $\vec{K}_{\mathbb{N}}$ and prove a variety of results relating to the upper density of paths in $\vec{K}_{\mathbb{N}}$.

Let $P = v_1v_2\ldots$ be a path in $\vec{K}_{\mathbb{N}}$. We say that P is a *directed path* if every edge in P is oriented in the same direction. If v_iv_{i+1} is an edge of P for each i, then P is a *forward* path; otherwise P is a *backward* path. By the *length* of a path, we mean the number of edges in the path. DeBiasio and McKenney [3] recently proved the following result.

Theorem 1.1. *For every $\varepsilon > 0$, there exists a 2-colouring of $\vec{K}_{\mathbb{N}}$ such that every monochromatic directed path has upper density less than ε.*

DeBiasio and McKenney also asked the following natural question: does there exist a 2-colouring of $\vec{K}_{\mathbb{N}}$ in which every monochromatic directed path has upper density 0? In Section 2, we will give a positive answer to this question (taken from the manuscript [8] by the third author). We note that the same example was independently obtained by Jan Corsten [2].

Theorem 1.2. *There exists a 2-colouring of $\vec{K}_{\mathbb{N}}$ such that every monochromatic directed path has upper density 0.*

In light of this result, it is natural to ask under what conditions we can guarantee the existence of a monochromatic path of positive density. It is easy to see (from Ramsey's Theorem) that every r-colouring of $\vec{K}_{\mathbb{N}}$ contains a monochromatic directed path of infinite length. The third author observed in [8] that if one restricts the maximal length of directed paths in the first colour, then there must be monochromatic paths in the second colour with non-vanishing upper density. More generally, the authors proved the following sequence of results:

- In the manuscript [8], Guggiari shows that for any $(r+1)$-edge-colouring of $\vec{K}_{\mathbb{N}}$ for which there are no directed paths of length ℓ_i in colour i for any $i \in [r]$, there is a monochromatic directed path in colour $r+1$ with upper density at least $\prod_{i\in[r]}\frac{1}{\ell_i}$.
- Confirming and extending a conjecture by Guggiari, Bürger and Pitz showed in the manuscript [1] that under the same assumptions as above, the vertex set $\mathbb{N}$ can be covered by $\prod_{i\in[r]}\ell_i$ pairwise disjoint monochromatic directed paths in colour $r+1$.

In Section 3 of this paper, we will present a proof of the following statement, which nicely generalizes the results above. Write $\vec{K}$ for the complete symmetric digraph on a finite or (not-necessarily-countable) infinite number of vertices.

Theorem 1.3. *Let $c : E(\vec{K}) \to [r+1]$ be an edge-colouring of $\vec{K}$ for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$. Then there is a partition of $V(\vec{K})$ into $\prod_{i\in[r]}\ell_i$ complete symmetric digraphs monochromatic in colour $r+1$.*

In particular, for the countable directed complete digraph $\vec{K}_{\mathbb{N}}$, we reobtain the result, now with a much shorter proof, that for any $(r+1)$-edge-colouring of $\vec{K}_{\mathbb{N}}$ for which there are no monochromatic directed paths of length ℓ_i in colour i for any $i \in [r]$, the vertex set $\mathbb{N}$ can be covered by $\prod_{i\in[r]}\ell_i$ pairwise disjoint monochromatic directed paths in colour $r+1$.

It is not hard to see that this result is best possible: There are colourings of $\vec{K}_{\mathbb{N}}$, which we will call *cube colourings* for their geometric structure, that witness optimality of the above result, see Theorem 3.6. For example, it is not hard to see that any blue monochromatic path in the colouring of Fig. 1 has upper density at most $\frac{1}{6}$.

Extending the work of Guggiari in [8, Theorem 1.4], we will prove a stability version of Theorem 1.3 in Section 4 of this paper. This stability result, Theorem 4.2, says that any colouring $c : E(\vec{K}_{\mathbb{N}}) \to [r+1]$ for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$ and every directed path in colour $r+1$ has upper density at most $\prod_{i\in[r]}\frac{1}{\ell_i}$, must essentially be of the same cubic structure as the *cube colouring*, where the meaning of 'essentially' will be explained in Section 4.

2. An example where all monochromatic paths have density 0

Proof of Theorem 1.2. We colour the edges of $\vec{K}_{\mathbb{N}}$ in the following way. Let $m, n \in \mathbb{N}$ be distinct positive integers. Set $t = \min\{s \in \mathbb{N} : m \not\equiv n \bmod 2^s\}$. Exchanging m and n if necessary, we may assume that $m \equiv x \bmod 2^t$ where $x \in \{0, \ldots, 2^{t-1}-1\}$ and $n \equiv 2^{t-1} + x \bmod 2^t$. We colour (m, n) red and (n, m) blue.

[1] In fact, Rado proved that in any r-colouring of the edges of $\vec{K}_{\mathbb{N}}$ there is a partition of the vertices into at most r disjoint *anti-directed* paths of distinct colours. This implies the undirected version stated above.