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Note

Further identities on Catalan numbers

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ABSTRACT

Three summation formulae on the λ-extended Catalan numbers are established by means of hypergeometric series approach with one of them being provided a combinatorial proof through lattice path countings.

1. Introduction and motivation

Let $\mathbb{N}$ be the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. The Catalan numbers defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{with} \quad n \in \mathbb{N}_0 \tag{1}$$

are one of the most interesting sequences in classical combinatorics, which has more than fifty significant combinatorial interpretations [14, p. 220]. One slightly extended polynomial form in an indeterminate λ reads as

$$C_n^{\lambda} = \frac{\lambda}{2n+\lambda}\binom{2n+\lambda}{n} \qquad \text{with} \qquad n \in \mathbb{N}_0 \tag{2}$$

which counts, for $\lambda \in \mathbb{N}$, the lattice paths from the origin to $(n+\lambda, n)$ in the first quadrant remaining strictly below the main diagonal (after the departure) with horizontal and vertical steps of unit length (cf. Mohanty [12, Chapter 1]).

There exist numerous identities concerning the Catalan numbers in the mathematical literature (see [1,5,11] just for examples). Koshy [10, p. 322] recorded the following interesting one

$$\sum_{k\geq 0}(-1)^k\binom{n-k+1}{k}C_{n-k} = \delta_{n,0} \tag{3}$$

where $\delta_{i,j}$ denotes the Kronecker symbol with $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. Zhou and Chu [15] derived several extended polynomial identities. Three of them can be reproduced as follows:

$$\sum_{k\geq 0}(-1)^k\binom{n-k}{k}C_{n-k}^{\lambda} = \frac{\lambda}{\lambda+n}\binom{\lambda+n}{n}, \tag{4}$$

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$$\sum_{k\geq 0}(-1)^k\binom{n-k+\lambda}{k}C^{\lambda}_{n-k}=\delta_{n,0},\tag{5}$$

$$\sum_{k\geq 0}(-1)^k\binom{n-k+\varepsilon}{k}C_{n-k}=\delta_{n,0};\quad (1\leq\varepsilon\leq n).\tag{6}$$

Recently in determining explicitly the inverse for a triangular matrix of binomial entries, Qi–Zou–Guo [13] found some remarkable identities involving classical Catalan numbers, that motivate the author to explore further similar identities containing another integer parameter "λ". This will be done by the hypergeometric series approach. According to Bailey [3], the classical hypergeometric series reads as

$${}_{1+p}F_p\left[\begin{matrix}a_0,\ a_1,\ a_2,\ \cdots,\ a_p\\ b_1,\ b_2,\ \cdots,\ b_p\end{matrix}\bigg|\, z\right]=\sum_{k=0}^{\infty}\frac{(a_0)_k(a_1)_k(a_2)_k\cdots(a_p)_k}{k!\,(b_1)_k(b_2)_k\cdots(b_p)_k}z^k$$

where the shifted-factorial is given by

$$(x)_0=1\quad\text{and}\quad (x)_n=x(x+1)\cdots(x+n-1)\quad\text{for}\quad n\in\mathbb{N}$$

with its quotient form being abbreviated by

$$\left[\begin{matrix}a,b,\ldots,c\\ \alpha,\beta,\ldots,\gamma\end{matrix}\right]_n=\frac{(a)_n(b)_n\cdots(c)_n}{(\alpha)_n(\beta)_n\cdots(\gamma)_n}.$$

When one of numerator parameters $\{a_k\}$ is a negative integer, the series just displayed becomes terminating, which reduces to a polynomial in z. In addition, the series will be called balanced if the sum of its denominator parameters exceeds by one that of the numerator ones.

The rest of the paper will be organized as follows. In the next section, we shall prove, by making use of the hypergeometric summation formulae and contiguous relations, three main identities concerning the λ-extended Catalan numbers C^{λ}_k and two integer parameters m and n. As special cases, they recover one theorem and refine another, found recently by Qi–Zou–Guo [13]. Then in the third section, we shall construct a combinatorial proof for one of our main results through lattice path countings.

2. Main theorems and proofs

Recall the following formula for terminating hypergeometric ${}_4F_3$-series due to Andrews and Burge [2] (see also Chu–Wei [6, Proposition 2]):

$${}_4F_3\left[\begin{matrix}-\frac{m}{2}, & \frac{1-m}{2}, & a, & c-a\\ & -m, & \frac{1+c}{2}, & \frac{2+c}{2}\end{matrix}\bigg|\,1\right]=\frac{(a)_{m+1}-(c-a)_{m+1}}{(2a-c)(1+c)_m}$$

where it is assumed due to the presence of the negative integer "$-m$" as one of the denominator parameter, that the summation index k for ${}_4F_3$-series runs from 0 to $\lfloor\frac{m}{2}\rfloor$, the integer part of $m/2$. We introduce the two binomial number expressions, with the second one being obtained by removing the ultimate term from the first series when m is even:

$$\mathcal{F}_n(m,\lambda):={}_4F_3\left[\begin{matrix}-\frac{m}{2},\frac{1-m}{2},-n,-\lambda-n\\ -m,\frac{1-\lambda}{2}-n,\frac{2-\lambda}{2}-n\end{matrix}\bigg|\,1\right]=\frac{(\lambda+n-m)_{m+1}-(n-m)_{m+1}}{\lambda(\lambda+2n-m)_m},\tag{7}$$

$$\begin{aligned}\mathcal{F}^{*}_n(m,\lambda)&:={}_4F_3\left[\begin{matrix}-\frac{m}{2},\frac{1-m}{2},-n,-\lambda-n\\ -m,\frac{1-\lambda}{2}-n,\frac{2-\lambda}{2}-n\end{matrix}\bigg|\,1\right]-\frac{\chi(m)(-m)_m}{(1-\lambda-2n)_m}\left[\begin{matrix}-n,-\lambda-n\\ 1,-m\end{matrix}\right]_{\lfloor\frac{m}{2}\rfloor}\\ &=\frac{(\lambda+n-m)_{m+1}-(n-m)_{m+1}}{\lambda(\lambda+2n-m)_m}-(-1)^{\frac{m}{2}}\frac{(-n)_{\lfloor\frac{m}{2}\rfloor}(-\lambda-n)_{\lfloor\frac{m}{2}\rfloor}}{(1-\lambda-2n)_m}\chi(m);\end{aligned}\tag{8}$$

where $\chi(m)$ is equal to 0 and 1, respectively, for odd and even m.

As a warm up, we prove now a polynomial identity in the indeterminate λ.

Theorem 1 ($m,n\in\mathbb{N}_0$).

$$\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^k\binom{m-k}{k}C^{\lambda}_{n-k}=C^{\lambda}_n\times\mathcal{F}_n(m,\lambda)\ =\ \binom{\lambda+2n-m-1}{n}-\binom{\lambda+2n-m-1}{n-m-1}.$$