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Partial sum of matrix entries of representations of the symmetric group and its asymptotics

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ABSTRACT

Many aspects of the asymptotics of Plancherel distributed partitions have been studied in the past fifty years, in particular the limit shape, the distribution of the longest rows, connections with random matrix theory and characters of the representation matrices of the symmetric group. Regarding the latter, we extend a celebrated result of Kerov on the asymptotic of Plancherel distributed characters by studying partial trace and partial sum of a representation matrix. We decompose each of these objects into a main term and a reminder, and for each such a decomposition we prove a central limit theorem for the main term. We apply these results to prove a law of large numbers for the partial sum. Our main tool is the expansion of symmetric functions evaluated on Jucys–Murphy elements.

1. Introduction

Let λ be a partition of n, in short $\lambda \vdash n$, represented as a Young diagram in English notation. A filling of the boxes of λ with numbers from 1 to n, increasing towards the right and downwards, is called a *standard Young tableau*. We call $\dim\lambda$ the number of standard Young tableaux of shape λ. We fix n and we associate to each λ the probability $\frac{(\dim\lambda)^2}{n!}$, which defines the *Plancherel measure*.

Let us recall briefly three results for the study of the asymptotics of Plancherel distributed random partitions. They relate algebraic combinatorics, representation theory of the symmetric group, combinatorics of permutations, and random matrix theory.

1. The partitions of n index the irreducible representations of the symmetric group S_n. For each $\lambda \vdash n$ the dimension of the corresponding irreducible representation is $\dim\lambda$. A natural question concerns the asymptotics of the associated characters when λ is distributed with the Plancherel measure. A central limit theorem was proved with different techniques by Kerov, [12,15], and Hora, [10]: consider a Plancherel distributed partition $\lambda \vdash n$ and ρ a fixed partition of r for $r \leq n$; set $m_k(\rho)$ to be the number of parts of ρ which are equal to k, and $\hat{\chi}^{\lambda}_{(\rho,1,\dots,1)} = \chi^{\lambda}_{(\rho,1,\dots,1)}/\dim\lambda$ the renormalized character associated to λ calculated on a permutation of cycle type $(\rho, 1, \dots, 1)$. Then for $n \to \infty$
$$n^{\frac{|\rho|-m_1(\rho)}{2}}\hat{\chi}^{\lambda}_{(\rho,1,\dots,1)} \xrightarrow{d} \prod_{k\geq 2} k^{m_k(\rho)/2}\mathcal{H}_{m_k(\rho)}(\xi_k)\,, \tag{1}$$
where $\mathcal{H}_m(x)$ is the mth Hermite polynomial, $\{\xi_k\}_{k\geq 2}$ are i.i.d. standard Gaussian variables, and $\xrightarrow{d}$ means convergence in distribution for $n \to \infty$.

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2. The Robinson–Schensted–Knuth algorithm allows us to interpret the longest increasing subsequence of a uniform random permutation as the first row of a Plancherel distributed partition. This motivates the study of the shape of a random partition. A limit shape result was proved independently by Kerov and Vershik [17] and by Logan and Shepp [18], then extended to a central limit theorem by Kerov, [12]. For an extensive introduction on the topic, see the book by Romik [21].
3. More recently it was proved that, after rescaling, the limiting distribution of the longest k rows of a Plancherel distributed partition λ coincides with the limit distribution of the properly rescaled k largest eigenvalues of a random Hermitian matrix taken from the Gaussian Unitary Ensemble. See for example the paper of Borodin, Okounkov and Olshanski [3] and references therein. Such similarities also occur for fluctuations of linear statistics, see the article of Ivanov and Olshanski [12].

In the aftermath of Kerov's result (1), a natural step in the study of the characters of the symmetric group is to look at the representation matrix rather than just the trace. We consider thus, for a real valued matrix A of dimension N and $u \in [0, 1]$, the partial trace and partial sum defined, respectively, as

$$PT_u(A) := \sum_{i \le uN} \frac{A_{i,i}}{N}, \qquad PS_u(A) := \sum_{i,j \le uN} \frac{A_{i,j}}{N}.$$

We study these values for representation matrices of S_n: let λ be a partition of n and σ a permutation in S_r, $r \le n$; we consider σ as a permutation of S_n in which the points $r+1, \dots, n$ are fixed points. We call π^λ the irreducible representation of S_n associated to λ. Thus, $\pi^\lambda(\sigma)$ is a square matrix of dimension $\dim\lambda$ whose entries are complex numbers. We study the values of $PT_u\left(\pi^\lambda(\sigma)\right)$ and $PS_u\left(\pi^\lambda(\sigma)\right)$ as functions of λ. In particular, we consider λ a random partition distributed with the Plancherel measure, and we study the random functions $PT_u\left(\pi^\lambda(\sigma)\right)$ and $PS_u\left(\pi^\lambda(\sigma)\right)$ when n grows. These partial sums are obviously not invariant by isomorphisms of representations, hence we consider an explicit natural construction of irreducible representations (the Young's orthogonal representation, which will be defined in Section 2.3). We define *subpartitions* μ_j of λ, denoted $\mu_j \nearrow \lambda$, the partitions of $n-1$ obtained from λ by removing one box. The orthogonal representation (and the right choice for the order of the basis elements) allows us to write the representation matrix $\pi^\lambda(\sigma)$ as a direct sum

$$\pi^\lambda(\sigma) = \bigoplus_{\mu \nearrow \lambda} \pi^\mu(\sigma)$$

if $\sigma \in S_r$ with $r \le n-1$. This property will be proven in Proposition 2.8. We deduce a decomposition of the partial trace and partial sum of a representation matrix: we will show that there exist $\bar{j} \in \mathbb{N}$ and $\bar{u} \in [0, 1]$ such that

$$PT_u^\lambda(\sigma) := PT_u\left(\pi^\lambda(\sigma)\right) = \sum_{j<\bar{j}} \frac{\dim\mu_j}{\dim\lambda} PT_1^{\mu_j}(\sigma) + \frac{\dim\mu_{\bar{j}}}{\dim\lambda} PT_{\bar{u}}^{\mu_{\bar{j}}}(\sigma), \tag{2}$$

$$PS_u^\lambda(\sigma) := PS_u\left(\pi^\lambda(\sigma)\right) = \sum_{j<\bar{j}} \frac{\dim\mu_j}{\dim\lambda} PS_1^{\mu_j}(\sigma) + \frac{\dim\mu_{\bar{j}}}{\dim\lambda} PS_{\bar{u}}^{\mu_{\bar{j}}}(\sigma). \tag{3}$$

Here $PT_1^{\mu_j}(\sigma) = \hat{\chi}^{\mu_j}(\sigma)$, while

$$PS_1^\lambda(\sigma) = \sum_{i \le \dim\lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim\lambda} =: TS^\lambda(\sigma)$$

is the *total sum* of the matrix $\pi^\lambda(\sigma)$.

In the first (resp. the second) decomposition we call the first term the *main term of the partial trace* $MT_u^\lambda(\sigma)$ (resp. *main term of the partial sum* $MS_u^\lambda(\sigma)$) and the second the *remainder for the partial trace* $RT_u^\lambda(\sigma)$ (resp. the *remainder for the partial sum* $RS_u^\lambda(\sigma)$). The decompositions show that the behavior of partial trace and partial sum depend on, respectively, total trace and total sum.

We consider Plancherel distributed partitions $\lambda \vdash n$. As recalled in Eq. (1), a central limit theorem for total traces, or characters, is well known; we prove a central limit theorem for the total sum (Theorem 4.2): to each $\sigma \in S_r$ we associate the two values

$$m_\sigma := \mathbb{E}_{\mathrm{Pl}}^r\left[TS^\nu(\sigma)\right] \qquad \text{and} \qquad v_\sigma := \binom{r}{2} \mathbb{E}_{\mathrm{Pl}}^r\left[\hat{\chi}^\nu_{(2,1,\dots,1)} TS^\nu(\sigma)\right],$$

where $\mathbb{E}_{\mathrm{Pl}}^r[X^\nu]$ is the average of the random variable X^ν considered with the Plancherel measure $(\dim\nu)^2/r!$ for $\nu \vdash r$. Then

Theorem 1.1. *Fix $\sigma \in S_r$ and let $n \ge r$. Consider $\lambda \vdash n$ a random partition distributed with the Plancherel measure, so that $TS^\lambda(\sigma)$ is a random function on the space of partitions of n. Then, for $n \to \infty$*

$$n \cdot \left(TS^\lambda(\sigma) - m_\sigma\right) \xrightarrow{d} \mathcal{N}\left(0, 2v_\sigma^2\right),$$

where $\mathcal{N}\left(0, 2v_\sigma^2\right)$ is a normal random variable of variance $2v_\sigma^2$ (provided that $v_\sigma \neq 0$t) and $\xrightarrow{d}$ means convergence in distribution.