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Note

# On the maximum size of connected hypergraphs without a path of given length

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ABSTRACT

In this note we asymptotically determine the maximum number of hyperedges possible in an $r$-uniform, connected $n$-vertex hypergraph without a Berge path of length $k$, as $n$ and $k$ tend to infinity. We show that, unlike in the graph case, the multiplicative constant is smaller with the assumption of connectivity.

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## 1. Introduction

Let $P_k$ denote a path consisting of $k$ edges in a graph $G$. There are several notions of paths in hypergraphs, the most basic of which is due to Berge. A Berge path of length $k$ is a set of $k+1$ distinct vertices $v_1, v_2, \ldots, v_{k+1}$ and $k$ distinct hyperedges $h_1, h_2, \ldots, h_k$ such that for $1 \leq i \leq k$, $v_i, v_{i+1} \in h_i$. A Berge path is also denoted simply as $P_k$, and the vertices $v_i$ are called basic vertices. If $v_1 = v$ and $v_{k+1} = w$, then we call the Berge path a Berge $v$-$w$-path. A hypergraph $\mathcal{H}$ is called connected if for any $v \in V(\mathcal{H})$ and $w \in V(\mathcal{H})$ there is a Berge $v$-$w$-path.

A classical result of Erdős and Gallai [3] asserts that

**Theorem 1** (*Erdős–Gallai*). *Let $G$ be a graph on $n$ vertices not containing $P_k$ as a subgraph, then*

$$|E(G)| \leq \frac{(k-1)n}{2}.$$

In fact, Erdős and Gallai deduced this result as a corollary of the following stronger result about cycles,

**Theorem 2** (*Erdős–Gallai*). *Let $G$ be a graph on $n$ vertices with no cycle of length at least $k$, then*

$$|E(G)| \leq \frac{(k-1)(n-1)}{2}.$$

Kopylov [5] and later Balister, Győri, Lehel and Schelp [1] determined the maximum number of edges possible in a connected $P_k$-free graph.

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**Theorem 3.** *Let $G$ be a connected $n$-vertex graph with no $P_k$, $n > k \geq 3$. Then $|E(G)|$ is bounded above by*

$$\max\left\{\binom{k-1}{2}+n-k+1,\binom{\lceil\frac{k+1}{2}\rceil}{2}+\left\lfloor\frac{k-1}{2}\right\rfloor\left(n-\left\lceil\frac{k+1}{2}\right\rceil\right)\right\}.$$

Observe that, although the upper bound is smaller in the connected case, it is nonetheless the same asymptotically. Balister, Győri, Lehel and Schelp also determined the extremal cases.

**Definition 1.** The graph $H_{n,k,a}$ consists of 3 disjoint vertex sets $A, B, C$ with $|A| = a$, $|B| = n - k + a$ and $|C| = k - 2a$. $H_{n,k,a}$ contains all edges in $A \cup C$ and all edges between $A$ and $B$. $B$ is taken to be an independent set. The number of $s$-cliques in this graph is

$$f_s(n,k,a)=\binom{k-a}{s}+(n-k+a)\binom{a}{s-1}.$$

The upper bound of Theorem 3 is attained for the graph $H_{n,k,1}$ or $H_{n,k,\lfloor\frac{k-1}{2}\rfloor}$.

We now mention some recent results of Luo [6] which will be essential in our proof. Let $N_s(G)$ denote the number of $s$-vertex cliques in the graph $G$.

**Theorem 4** (*Luo*)**.** *Let $n - 1 \geq k \geq 4$. Let $G$ be a connected $n$-vertex graph with no $P_k$, then*

$$N_s(G) \leq \max\{f_s(n,k,\lfloor(k-1)/2\rfloor), f_s(n,k,1)\}.$$

As a corollary, she also showed

**Corollary 1** (*Luo*)**.** *Let $n \geq k \geq 3$. Assume that $G$ is an $n$-vertex graph with no cycle of length $k$ or more, then*

$$N_s(G) \leq \frac{n-1}{k-2}\binom{k-1}{s}.$$

Győri, Katona and Lemons [4] initiated the study of Berge $P_k$-free hypergraphs. They proved

**Theorem 5** (*Győri–Katona–Lemons*)**.** *Let $\mathcal{H}$ be an $r$-uniform hypergraph with no Berge path of length $k$. If $k > r + 1 > 3$, we have*

$$|E(\mathcal{H})| \leq \frac{n}{k}\binom{k}{r}.$$

*If $r \geq k > 2$, we have*

$$|E(\mathcal{H})| \leq \frac{n(k-1)}{r+1}.$$

*Moreover, in the first case, the bound is sharp (for infinitely many $n$) by taking vertex-disjoint cliques of size $k$. In the second case, the bound is sharp (for infinitely many $n$) by taking disjoint sets of size $r + 1$, each containing $k - 1$ hyperedges.*

The case when $k = r + 1$ was settled later [2]:

**Theorem 6** (*Davoodi–Győri–Methuku–Tompkins*)**.** *Let $\mathcal{H}$ be an $n$-vertex $r$-uniform hypergraph. If $|E(\mathcal{H})| > n$, then $\mathcal{H}$ contains a Berge path of length at least $r + 1$.*

Our main result is the asymptotic upper bound for the connected version of Theorem 5, as $n$ and $k$ tend to infinity.

**Theorem 7.** *Let $\mathcal{H}_{n,k}$ be a largest $r$-uniform connected $n$-vertex hypergraph with no Berge path of length $k$, then*

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{|E(\mathcal{H}_{n,k})|}{k^{r-1}n}=\frac{1}{2^{r-1}(r-1)!}.$$

A construction yielding the bound in Theorem 7 is given by partitioning an $n$-vertex set into two classes $A$, of size $\lfloor\frac{k-1}{2}\rfloor$, and $B$, of size $n-\lfloor\frac{k-1}{2}\rfloor$ and taking $X \cup \{y\}$ as a hyperedge for every $(r-1)$-element subset $X$ of $A$ and every element $y \in B$. This hypergraph has no Berge $P_k$ as we could have at most $\lfloor\frac{k-1}{2}\rfloor$ basic vertices in $A$ and $\lfloor\frac{k-1}{2}\rfloor + 1$ basic vertices in $B$, thus yielding less than the required $k + 1$ basic vertices.

Observe that in Theorem 5 the corresponding limiting value of the constant factor is $\frac{1}{r!}$ which is $\frac{2^{r-1}}{r}$ times larger than in the connected case (the ideas of the proof of Theorem 7 can also be used to give a simple proof of the following weaker version of Theorem 5: The limiting value of the constant factor in Theorem 5 is $\frac{1}{r!}$).

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