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Countable linear orders with disjoint infinite intervals are mutually orthogonal

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ABSTRACT

Two linear orderings of a same set are *perpendicular* if the only self-mappings of this set that preserve them both are the identity and the constant mappings. Two linear orderings are *orthogonal* if they are isomorphic to two perpendicular linear orderings. We show that two countable linear orderings are orthogonal as soon as each one has two disjoint infinite intervals. From this and previously known results it follows in particular that each countably infinite linear ordering is orthogonal to itself.

1. Introduction and presentation of the results

1.1. Background

In this text we call *chain*, or *linear order*, every linearly ordered set $\mathfrak{C} = (C, \leq)$; thus $\leq$ is a *linear ordering* of the *vertex set* C. Given two chains $\mathfrak{C}_1 = (C_1, \leq_1)$ and $\mathfrak{C}_2 = (C_2, \leq_2)$, a mapping $f : C_1 \to C_2$ is called a *homomorphism* if it is *order preserving*, *i.e.* if $x \leq_1 y$ implies $f(x) \leq_2 f(y)$; this is an *embedding* if in addition it is one-to-one, or if it is *strictly increasing*, *i.e.* if $x <_1 y$ implies $f(x) <_2 f(y)$. Say that a chain $\mathfrak{C}_2$ *embeds* $\mathfrak{C}_1$, or that $\mathfrak{C}_1$ *embeds in* $\mathfrak{C}_2$, and write $\mathfrak{C}_1 \hookrightarrow \mathfrak{C}_2$, if there exists an embedding from $\mathfrak{C}_1$ into $\mathfrak{C}_2$. When the two chains are equal, a homomorphism is called an *endomorphism* and an embedding is called a *self-embedding*.

Definition 1.1 (*Perpendicular Orderings*)**.** Two orderings $\leqslant$ and $\preccurlyeq$ of a same set are *perpendicular* if every self-mapping of this set that preserves them both is constant or is the identity.

1.1.1. Finite perpendicular linear orderings

The notion of perpendicularity was introduced by Demetrovics et al. [6]. It appears naturally when studying the lattice of clones on a set (see also [11,14]). The first examples of perpendicular orderings were given by Demetrovics and Ronyai [7] (see also [15]). Miyakawa et al. [13] (see also [18]) observed that a linear ordering on a finite n-element set has a perpendicular

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linear ordering provided that $n \neq 3$, and they proved that the proportion of those converges to e^{-2} as n tends to infinity. Their counting argument was based on the fact that two linear orderings of the same *finite* set are perpendicular if and only if their common intervals are trivial (*i.e.* are the empty set, the singletons and the vertex set).

This notion reappeared in recent years in the setting of permutations of finite sets. Indeed the linear orderings of the set $[n] := \{1, \ldots, n\}$ identify with the permutations of that set (*cf.* [4]): to each permutation φ of $[n]$ there corresponds the linear ordering of $[n]$ given by $x \leq_\varphi y \Leftrightarrow \varphi(x) \leq \varphi(y)$, and then the *bichain* $\mathfrak{B}_\varphi := ([n], \leq, \leq_\varphi)$. Of particular interest are those bichains for which $\leq_\varphi$ is perpendicular to the natural ordering $\leq$ (see [1–3,9,12]).

1.1.2. Orthogonal types of countable linear orderings

Since two linear orderings of a same infinite set may fail to be isomorphic, one is lead to the following definition.

Definition 1.2 (*Orthogonal Types of Ordered Sets*)**.** Two isomorphy types of orderings τ and θ are *orthogonal* if there exists a pair of perpendicular orderings of these types . In this case, we write $\tau \perp \theta$. An isomorphy type is *self-orthogonal* if it is orthogonal to itself.

We also say that two ordered set $(C, \leqslant)$ and $(D, \preccurlyeq)$ are *orthogonal* if their isomorphy types are orthogonal. An ordered set $\mathfrak{C} = (C, \leqslant)$ is *orthogonal* to a type θ if the type of $\mathfrak{C}$ is orthogonal to θ, equivalently, if there is an ordering $\preccurlyeq$ of type θ on C that is perpendicular to $\leqslant$.

For instance, notice that, with these definitions, the usual ordering of the chain $\mathbb{N}$ of natural numbers is not perpendicular to itself (*cf.* Lemma 2.4), but its type is self-orthogonal according to (1.4) below.

Notation 1.1 (*The Types ω, ζ, η and ω^**)**.** As usual, ω denotes the first infinite ordinal, that is the type of the chain $(\mathbb{N}, \leq)$ of natural numbers. The type of the chain $(\mathbb{Z}, \leq)$ of relative integers is denoted by ζ. The order type of the chain $(\mathbb{Q}, \leq)$ of rational numbers is denoted by η; this is the type of any denumerable dense linear order (*i.e.* with no consecutive vertices) without least nor greatest element. If τ is the type of a chain $(C, \leqslant)$ then the type of the reversed chain $(C, \geqslant)$ is denoted by τ^*. Thus ω^* is the type of the chain $(\mathbb{Z}_-, \leq)$ of negative integers.

Note that, since an order and the reversed order have the same endomorphisms, for any two types θ and τ:

$$\tau \perp \theta \Longleftrightarrow \tau^* \perp \theta. \tag{1.1}$$

In this text, we mainly consider countable chains. Orthogonality of countable chains has been studied in [10,17]. Let us recall what is already known on the matter. Notice that in this text *denumerable* means *countably infinite*, and that *countable* means *at most countable*.

On the one hand, pairs of orthogonal denumerable ordinals have been characterized in [10]:

(1) Any two countable ordinals greater than ω are orthogonal [10]:

$$\omega < \alpha, \beta < \omega_1 \Longrightarrow \alpha \perp \beta \tag{1.2}$$

(2) A denumerable ordinal is orthogonal to ω *if and only if* the least term of its Cantor decomposition is a finite power of ω; in other words [10]:

$$\forall \alpha < \omega_1 \, (\alpha \perp \omega \Longleftrightarrow \omega \leq \alpha < \omega \cdot \alpha) \tag{1.3}$$

In particular:

$$\omega \perp \omega \tag{1.4}$$

On the other hand, orthogonality to the order type η of rational numbers has been investigated in [17]:

(3) The order type η of rational numbers is self-orthogonal [17]: $\eta \perp \eta$.
(4) No infinite ordinal less than $\omega + \omega$ is orthogonal to η [17]:

$$\alpha < \omega + \omega \Longrightarrow \alpha \not\perp \eta \tag{1.5}$$

(5) Every countable ordinal greater than or equal to ω^2 is orthogonal to η [17]:

$$\omega^2 \leq \alpha < \omega_1 \Longrightarrow \alpha \perp \eta.$$

In particular the following question, which was the original motivation of the present work, was left open in [17]: *Is η orthogonal to any ordinal α such $\omega + \omega \leq \alpha < \omega^2$?* From our main result (Theorem 1.1 below), it follows that η is orthogonal to each such ordinal, which answers the question. Hence: *an ordinal is orthogonal to η if and only if it is greater than or equal to $\omega + \omega$* (see Corollary 1.1-(1) and Proposition 3.2 below).