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Note

# Self-complementary magic squares of doubly even orders

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ABSTRACT

A magic square $M$ in which the entries consist of consecutive integers from $1, 2, \ldots, n^2$ is said to be *self-complementary of order n* if the resulting square obtained from $M$ by replacing each entry $i$ by $n^2 + 1 - i$ is equivalent to $M$ (under rotation or reflection). We present a new construction for self-complementary magic squares of order $n$ for each $n \geq 4$, where $n$ is a multiple of 4.



## 1. Introduction

A *magic square of order n* is a square array of integers from $1, 2, \ldots, n^2$ such that the sum of entries in each row, column and diagonal is the same number, which is called the *magic sum* of the square. It is easy to see that the magic sum of an $n$th order magic square is $n(n^2 + 1)/2$.

Magic squares are probably among the earliest combinatorial objects known. A number of unsolved research problems on magic squares are available in [1]. Here we present an interesting construction for a special kind of magic squares that contain certain symmetry.

Suppose $M = (a_{i,j})$ is a magic square of order $n$. We say that $M$ is *symmetrical* if $a_{i,j} + a_{n+1-i,n+1-j} = n^2 + 1$ for all $1 \leq i, j \leq n$. Note that if $M$ is symmetrical of order $n$, then $M + \sigma(M) = (n^2 + 1)J_n$ where $\sigma(M)$ is the 180-degree clockwise rotation on $M$ and $J_n$ is the $n \times n$ matrix with every entry equals 1. In view of this rotational property, a symmetrical magic square is also called a *ro-symmetrical* magic square in [2].

Let $\pi(M)$ denote the 180-degree reflection on $M$ along the central vertical of $M$. Then we say that $M = (a_{i,j})$ is *ref-symmetrical* if $M + \pi(M) = (n^2 + 1)J_n$. In this case we have $a_{i,j} + a_{i,n+1-j} = n^2 + 1$ for all $1 \leq i, j \leq n$. Equivalently we could take $\pi(M)$ to be the 180-degree reflection on the central horizontal of $M$, in which case we have $a_{i,j} + a_{n+1-i,j} = n^2 + 1$ for all $1 \leq i, j \leq n$ instead.

By replacing every entry $x$ in $M$ with $n^2 + 1 - x$ we obtain the *complement* of $M$ which is also a magic square of order $n$. If $M$ is equivalent to its complement (under rotation or reflection), we say that $M$ is *self-complementary*. Incidentally, the Lo-Shu, one of the earliest combinatorial designs (see [3]) is an example of self-complementary magic square of order 3. The two magic squares of order 4 depicted in Fig. 1 are both self-complementary. Here $M_1$ is ro-symmetrical while $M_2$ is ref-symmetrical.

Self-complementary magic squares were investigated in [2] where a characterization for self-complementary magic squares was presented. It was shown that (i) if $n$ is odd, then $M$ is self-complementary if and only if $M$ is ro-symmetrical, and that (ii) if $n$ is even, then $M$ is self-complementary if and only if either $M$ is ref-symmetrical or else $M$ is ro-symmetrical; in the case that $M$ is ro-symmetrical, $n \equiv 0 \pmod 4$.

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$M_1 =$

| | | | |
|---|---|---|---|
| 16 | 2 | 3 | 13 |
| 5 | 11 | 10 | 8 |
| 9 | 7 | 6 | 12 |
| 4 | 14 | 15 | 1 |

$M_2 =$

| | | | |
|---|---|---|---|
| 16 | 11 | 6 | 1 |
| 2 | 5 | 12 | 15 |
| 3 | 8 | 9 | 14 |
| 13 | 10 | 7 | 4 |

**Fig. 1.** Self-complementary magic squares of order 4.

While there are several known methods of constructions for ro-symmetrical magic squares, not much is known about the construction for ref-symmetrical magic squares except for the construction given in [2]. The purpose of this note is to present a new method of constructing ref-symmetrical magic squares of doubly even order (see Theorem 1). An interesting part of this construction is that it converts a ro-symmetrical magic square $M$ of doubly even order (obtained from a well-known construction called the Generalized Doubly Even Method (*GDEM*)) into a ref-symmetrical magic square of the same order. Basically $M$ is partitioned into $4 \times 4$ sub-squares followed by an operation on these sub-squares which are then used as ingredient sub-squares to form a ref-symmetrical magic square (see Section 2).

The *GDEM* construction is available in [4] (p. 199–200) and [3] (p. 527, Section 34.21). For ease of reference the *GDEM* construction is given below.

**Generalized Doubly Even Method:**

First, starting with the first row of the $n \times n$ square (where $n \equiv 0 \pmod 4$), fill the cells with $1, 2, \ldots, n^2$ in the natural order. Next, partition the square into $(n/4)^2$ $4 \times 4$ sub-squares. Finally, replace each integer $x$ which occurs in the main or back diagonal of each $4 \times 4$ sub-square with $n^2 + 1 - x$.

The magic square $M_1$ as depicted in Fig. 1 is obtained by using the *GDEM* construction with $n = 4$. For $n = 8$, the *GDEM* construction yields the following ro-symmetrical magic square.

$M_3 =$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 64 | 2 | 3 | 61 | 60 | 6 | 7 | 57 |
| 9 | 55 | 54 | 12 | 13 | 51 | 50 | 16 |
| 17 | 47 | 46 | 20 | 21 | 43 | 42 | 24 |
| 40 | 26 | 27 | 37 | 36 | 30 | 31 | 33 |
| 32 | 34 | 35 | 29 | 28 | 38 | 39 | 25 |
| 41 | 23 | 22 | 44 | 45 | 19 | 18 | 48 |
| 49 | 15 | 14 | 52 | 53 | 11 | 10 | 56 |
| 8 | 58 | 59 | 5 | 4 | 62 | 63 | 1 |

## 2. The construction

Let $A = (a_{i,j})$ denote a $4 \times 4$ matrix and define

$$\varphi(A) = \begin{bmatrix} a_{1,1} & a_{2,2} & a_{3,3} & a_{4,4} \\ a_{1,2} & a_{2,1} & a_{3,4} & a_{4,3} \\ a_{1,3} & a_{2,4} & a_{3,1} & a_{4,2} \\ a_{1,4} & a_{2,3} & a_{3,2} & a_{4,1} \end{bmatrix}.$$

Note that (i) the main diagonal sum of $\varphi(A)$ is equal to the first column sum of $A$, (ii) the back diagonal sum of $\varphi(A)$ is equal to the last column sum of $A$, and (iii) the $i$th column sum of $\varphi(A)$ is equal to the $i$th row sum of $A$.

Suppose that $B = (b_{i,j})$ is a $4 \times 4$ matrix. Define

$$\alpha(A, B) = \begin{bmatrix} a_{1,1} & a_{2,1} & b_{3,4} & b_{4,4} \\ a_{1,2} & a_{2,2} & b_{3,3} & b_{4,3} \\ a_{1,3} & a_{2,3} & b_{3,2} & b_{4,2} \\ a_{1,4} & a_{2,4} & b_{3,1} & b_{4,1} \end{bmatrix} \quad \textit{and} \quad \beta(A, B) = \begin{bmatrix} a_{3,1} & a_{4,1} & b_{1,4} & b_{2,4} \\ a_{3,2} & a_{4,2} & b_{1,3} & b_{2,3} \\ a_{3,3} & a_{4,3} & b_{1,2} & b_{2,2} \\ a_{3,4} & a_{4,4} & b_{1,1} & b_{2,1} \end{bmatrix}.$$

Note that the $i$th column sum of $\alpha(A, B)$ is equal to the $i$th row sum of $A$ for $i = 1, 2$ and is equal to the $i$th row sum of $B$ for $i = 3, 4$. Likewise, the $i$th column sum of $\beta(A, B)$ is equal to the $(i + 2)$th row sum of $A$ if $i = 1, 2$, and is equal to the $(i - 2)$-th row sum of $B$ if $i = 3, 4$.

Let $M$ be the ro-symmetrical magic square of order $n$ obtained by using the *GDEM* construction. Partition $M$ into $4 \times 4$ sub-matrices $A_{i,j}$ where $1 \le i, j \le n/4$.

(I) Suppose $n \equiv 0 \pmod 8$.

Let $\varphi(M)$ denote the $n \times n$ matrix whose $(i, j)$-entry is the $4 \times 4$ matrix

(i) $\varphi(A_{j,\,i})$ if $1 \le i \le n/4$ and $1 \le j \le n/8$,

(ii) $\varphi(A_{j,\,n/4+1-i})$ if $1 \le i \le n/4$ and $n/8 + 1 \le j \le n/4$.