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Degree sum and hamiltonian-connected line graphs

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ABSTRACT

In 1984, Bauer proposed the problems of determining best possible sufficient conditions on the vertex degrees of a simple graph (or a simple bipartite graph, or a simple triangle-free graph, respectively) G to ensure that its line graph $L(G)$ is hamiltonian. We investigate the problems of determining best possible sufficient conditions on the vertex degrees of a simple graph G to ensure that its line graph $L(G)$ is hamiltonian-connected, and prove the following.

(i) For any real numbers a, b with $0 < a < 1$, there exists a finite family $\mathcal{F}(a, b)$ such that for any connected simple graph G on n vertices, if $d_G(u)+d_G(v) \geq an+b$ for any $u, v \in V(G)$ with $uv \notin E(G)$, then either $L(G)$ is hamiltonian-connected, or $\kappa(L(G)) \leq 2$, or $L(G)$ is not hamiltonian-connected, $\kappa(L(G)) \geq 3$ and G is contractible to a member in $\mathcal{F}(a, b)$.

(ii) Let G be a connected simple graph on n vertices. If $d_G(u) + d_G(v) \geq \frac{n}{4} - 2$ for any $u, v \in V(G)$ with $uv \notin E(G)$, then for sufficiently large n, either $L(G)$ is hamiltonian-connected, or $\kappa(L(G)) \leq 2$, or $L(G)$ is not hamiltonian-connected, $\kappa(L(G)) \geq 3$ and G is contractible to W_8, the Wagner graph.

(iii) Let G be a connected simple triangle-free (or bipartite) graph on n vertices. If $d_G(u) + d_G(v) \geq \frac{n}{8}$ for any $u, v \in V(G)$ with $uv \notin E(G)$, then for sufficiently large n, either $L(G)$ is hamiltonian-connected, or $\kappa(L(G)) \leq 2$, or $L(G)$ is not hamiltonian-connected, $\kappa(L(G)) \geq 3$ and G is contractible to W_8, the Wagner graph.

1. The problem

We consider finite loopless graphs but multiple edges are permitted and follow [4] for undefined terms and notation. As in [4], $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and edge-connectivity of a graph G, respectively. We define $\kappa'(K_1) = \infty$. An edge cut with size k is called a k-edge-cut. For an integer $i \geq 0$, we define $V_i(G) = \{v \in V(G) \mid d_G(v) = i\}$ and $d_i(G) = |V_i(G)|$. For vertices $u, v \in V(G)$, a (u, v)-path (a (u, v)-trail, respectively) is a path (a trail, respectively) from u to v. A graph is hamiltonian if it has a spanning cycle, and is **hamiltonian-connected** if for any distinct vertices u and v, G contains a spanning (u, v)-path. It is well known that every hamiltonian-connected graph must be 3-connected. The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common.

If $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If H is a subgraph of G, we write G/H for $G/E(H)$. If H is a connected subgraph of G and v_H is the vertex in G/H onto which H is contracted, then H is the **preimage** of v_H and is denoted by $PI_G(v_H)$.

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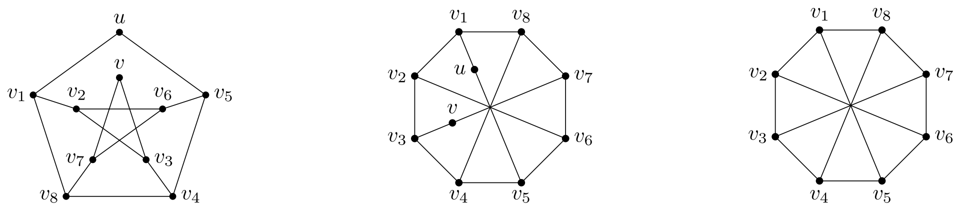

(a) $P(10) - uv$. (b) $P(10) - uv$. (c) W_8.

Fig. 1. Wagner graph and related graphs.

In [1,2], Bauer proposed the problems of determining best possible sufficient conditions on the vertex degrees of a simple graph (or a simple bipartite graph, or a simple triangle-free graph, respectively) G to ensure that its line graph $L(G)$ is hamiltonian. These problems have been settled by Catlin [6] and Lai [15]. Similar problems are considered in this paper. We seek best possible sufficient degree conditions of a simple graph G to assure that $L(G)$ is hamiltonian-connected. The graph W_8 depicted in Fig. 1(c) is called the **Wagner graph**. Our main results in this paper are the following.

Theorem 1.1. *Let $n \geq 3$ be an integer. For any real numbers a, b with $0 < a < 1$, there exists a family $\mathcal{F}(a, b)$ of finitely many graphs each of which has a non-hamiltonian-connected line graph, such that for any connected simple graph G on n vertices, if*

$$d_G(u) + d_G(v) \geq an + b \text{ for any } u, v \in V(G) \text{ with } uv \notin E(G), \tag{1.1}$$

then exactly one of the following must hold:
(i) *$L(G)$ is hamiltonian-connected;*
(ii) *$\kappa(L(G)) \leq 2$;*
(iii) *$L(G)$ is not hamiltonian-connected, $\kappa(L(G)) \geq 3$ and G is contractible to a member in $\mathcal{F}(a, b)$.*

Theorem 1.2. *Let $n \geq 3$ be an integer, and G be a connected simple graph on n vertices. If*

$$d_G(u) + d_G(v) \geq \frac{n}{4} - 2 \text{ for any } u, v \in V(G) \text{ with } uv \notin E(G), \tag{1.2}$$

then for sufficiently large n, exactly one of the following must hold:
(i) *$L(G)$ is hamiltonian-connected;*
(ii) *$\kappa(L(G)) \leq 2$;*
(iii) *$L(G)$ is not hamiltonian-connected, $\kappa(L(G)) \geq 3$ and G is contractible to W_8.*

Theorem 1.3. *Let G be a connected simple triangle-free (or bipartite) graph on n vertices. If*

$$d_G(u) + d_G(v) \geq \frac{n}{8} \text{ for any } u, v \in V(G) \text{ with } uv \notin E(G), \tag{1.3}$$

then for sufficiently large n, exactly one of the following must hold:
(i) *$L(G)$ is hamiltonian-connected;*
(ii) *$\kappa(L(G)) \leq 2$;*
(iii) *$L(G)$ is not hamiltonian-connected, $\kappa(L(G)) \geq 3$ and G is contractible to W_8.*

In the next section, we present our associate results and develop some needed tools. In Section 3, we assume the truth of the associate results to prove our main results on hamiltonian-connected line graphs. The proofs for our associate results will be given in the last section.

2. Strongly spanning trailable graphs

For a graph G, let $O(G)$ denote the set of odd degree vertices in G. A graph G is **eulerian** if G is connected with $O(G) = \emptyset$, and is **supereulerian** if G has a spanning eulerian subgraph. Supereulerian graphs are first introduced by Boesch, Suffel, and Tindell in [3], and are closely related to the study of hamiltonian line graphs. Catlin [7] presented the first survey on supereulerian graphs. Supplemented or updated surveys on supereulerian graphs can be found in [11,16].

A graph G is **collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H such that $O(H) = R$. If G is collapsible, then by definition with $R = \emptyset$, G is supereulerian and so $\kappa'(G) \geq 2$. In [6], Catlin showed that for any graph G, every vertex of G lies in a unique maximal collapsible subgraph of G. The **reduction** of G, denoted by G', is obtained from G by contracting all nontrivial maximal collapsible subgraphs of G. A graph is **reduced** if it is the reduction of some graph. As shown in [6], a reduced graph is simple.