Discrete Mathematics 341 (2018) 1036–1040

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

The asymptotic dimension of box spaces of virtually nilpotent groups

Thiebout Delabie *, Matthew C.H. Tointon

Université de Neuchâtel, Switzerland

ARTICLE INFO

Article history:
Received 13 June 2017
Received in revised form 8 December 2017
Accepted 13 December 2017
Available online 29 January 2018

Keywords:
Box space
Asymptotic dimension
Virtually nilpotent
Polynomial growth

ABSTRACT

We show that every box space of a virtually nilpotent group has asymptotic dimension equal to the Hirsch length of that group.

1. Introduction

The principal objects of study of this note are so-called *box spaces*. These are metric spaces formed by stitching together certain finite quotients of residually finite groups, and have been the subject of much recent research, not least thanks to their utility in constructing metric spaces with unusual properties. For example, box spaces can be used to construct expanders [10], as well as to construct groups without property *A* [1,9,11].

Of particular interest to us will be the so-called *asymptotic dimension* of a box space. Defined by Gromov [8], the asymptotic dimension is a coarse version of the topological dimension. There has been a fair amount of recent work on computing the asymptotic dimension of various box spaces, and this note represents a contribution to that body of work.

Given a finitely generated residually finite infinite group G, a *filtration* of G is a sequence $(N_n)_{n=1}^{\infty}$ of nested, normal, finite-index subgroups N_n of G such that $\bigcap_n N_n = \{e\}$.

Definition 1.1 (*Box Space*)**.** The *box space* $\square_{(N_i)}G$ of a finitely generated residually finite infinite group G with respect to a filtration (N_n) of G and a finite generating set S of G is the metric space on the disjoint union of the quotients $(G/N_n)_n$ in which the metric on each component G/N_n is the Cayley-graph metric induced by S, and the distance between $x \in G/N_m$ and $y \in G/N_n$ with $m \neq n$ is defined to be the sum of the diameters of G/N_m and G/N_n.

Now let X be a metric space. Given $R > 0$, a family $\mathcal{U}$ of subsets of X is said to be *R-disjoint* if the distance between every pair of distinct sets in $\mathcal{U}$ is at least R. Given $R, S > 0$, the (R, S)*-dimension* $(R, S)\text{-}\dim(X)$ of X is defined to be the least $n \in \mathbb{Z}$ such that there exist families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of X such that

- $\bigcup_{j=0}^{n} \mathcal{U}_j$ covers X,
- $\mathcal{U}_j$ is R-disjoint for every $j \in \{0, 1, \ldots, n\}$, and
- $\operatorname{diam}(U) \leq S$ for every $U \in \mathcal{U}_j$ and $j \in \{0, 1, \ldots, n\}$.

* Corresponding author.
E-mail addresses: thiebout.delabie@unine.ch (T. Delabie), mcht2@cam.ac.uk (M.C.H. Tointon).

https://doi.org/10.1016/j.disc.2017.12.009

Definition 1.2 (*Asymptotic Dimension*). The *asymptotic dimension* $\operatorname{asdim} X$ of a metric space X is defined to be the least $n \in \mathbb{Z}$ such that for every $R > 0$ there exists $S > 0$ such that (R, S)- $\dim(X) \leq n$.

It is known that a virtually polycyclic group with a Cayley-graph metric has finite asymptotic dimension. Indeed, given a virtually polycyclic group G we write $h(G)$ for the *Hirsch length* of G, which is to say the number of infinite factors in a normal polycyclic series of a finite-index polycyclic subgroup of G. Dranishnikov and Smith [5, Theorem 3.5] show that if G is residually finite and virtually polycyclic then

$$\operatorname{asdim} G = h(G). \tag{1.1}$$

It is not unreasonable to expect that *box spaces* of virtually polycyclic groups should also have finite asymptotic dimension, and indeed there have been some results in this direction. For example, Szabó, Wu and Zacharias [13] show that every finitely generated virtually nilpotent group has *some* box space with finite asymptotic dimension. Finn-Sell and Wu [6] show moreover that for certain box spaces of virtually polycyclic groups the asymptotic dimension of the box space is, like that of the group itself, equal to the Hirsch length of the group. These results are not ideal in their current form, as they rely on certain subgroup inclusions inducing coarse embeddings of the corresponding box spaces, a fact that does not hold in general due to [4, Theorem 4.9].

The main purpose of the present note is to clarify the situation and strengthen these results in the case of virtually nilpotent groups. Indeed, we show that if G is a finitely generated virtually nilpotent group then in fact *every* box space of G has asymptotic dimension equal to the Hirsch length of G, as follows.

Theorem 1.3. *Let G be a finitely generated residually finite virtually nilpotent group and let $(N_n)_n$ be a filtration of G. Then* $\operatorname{asdim} \square_{(N_n)} G = h(G)$.

The note is organised as follows. In Section 2 we present some basic facts about box spaces and about asymptotic dimension; in Section 3 we compute the asymptotic dimension of certain box spaces in terms of the asymptotic dimensions of the groups they are constructed from; and then finally, in Section 4, we bound the asymptotic dimension of box spaces of groups of polynomial growth in terms of the growth rate and deduce Theorem 1.3.

2. Background

In this section we collect together various results about box spaces and asymptotic dimension. Given a group G with a fixed finite generating set S we write $B_G(x, R)$ for the ball of radius R about the element $x \in G$ in the Cayley graph $\mathrm{Cay}(G, S)$.

Coarse disjoint unions and box spaces. Given a sequence $(X_n)_{n=1}^{\infty}$ of sets we write $\bigsqcup_n X_n$ for their disjoint union. If (X_n, d_n) and $(\bigsqcup_n X_n, d)$ are all metric spaces then $\bigsqcup_n X_n$ is said to be a *coarse disjoint union* of the X_n if

(1) for each n the metric d restricts to d_n on X_n,
(2) whenever $m \neq n$ the distance between X_m and X_n is at least $\operatorname{diam} X_m + \operatorname{diam} X_n$, and
(3) for every r the number of pairs m, n with $d(X_m, X_n) < r$ is finite.

Note that if $\operatorname{diam} X_i \to \infty$ then property (2) of this definition implies property (3).

Remark. The exact distance between the components X_n of a coarse disjoint union does not change the coarse equivalence class. Since asymptotic dimension is a coarse invariant [3, Proposition 22], neither does it change the asymptotic dimension.

In the introduction we defined a box space $\square_{(N_i)} G$ of a finitely generated residually finite group G with respect to a filtration (N_n) of G and a finite generating set S of G to be the disjoint union of the quotients $(G/N_n)_n$ with the metric that restricts to the Cayley-graph metric induced by S on each G/N_n, and such that for $x \in G/N_m$ and $y \in G/N_n$ with $m \neq n$ the distance between x and y is precisely $\operatorname{diam}(G/N_m) + \operatorname{diam}(G/N_n)$. Note in particular that $\square_{(N_i)} G$ is a coarse disjoint union of the G/N_n.

We also define the *full box space* $\square_f G$ of G, setting it to be the disjoint union of *all* finite quotients of G with the metric that restricts to the Cayley-graph metric induced by S on each quotient, and such that for $x \in G/N$ and $y \in G/N'$ with $N \neq N'$ the distance between x and y is precisely $\operatorname{diam}(G/N) + \operatorname{diam}(G/N')$.

The following standard result says that the components of a box space locally 'look like' the group.

Proposition 2.1. *Let G be a residually finite, finitely generated group and let $(N_i)_i$ be a filtration of G. Then there exists an increasing sequence $(i_k)_k$ such that for every $k \in \mathbb{N}$ the balls of radius k of G are isometric to the balls of radius k of G/N_i, where $i \geq i_k$.*

Proof. For a given k and large enough i we have $B_G(e, 2k) \cap N_i = \{1\}$, which in turn implies that $B_G(e, k)$ is isometric to $B_{G/N_i}(e, k)$. $\square$