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Note

Distant total irregularity strength of graphs via random vertex ordering

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ABSTRACT

Let $c : V \cup E \to \{1, 2, \dots, k\}$ be a (not necessarily proper) total colouring of a graph $G = (V, E)$ with maximum degree Δ. Two vertices $u, v \in V$ are *sum distinguished* if they differ with respect to sums of their incident colours, i.e. $c(u)+\sum_{e\ni u} c(e) \neq c(v)+\sum_{e\ni v} c(e)$. The least integer k admitting such colouring c under which every $u, v \in V$ at distance $1 \leq d(u, v) \leq r$ in G are sum distinguished is denoted by $\mathrm{ts}_r(G)$. Such graph invariants link the concept of the total vertex irregularity strength of graphs with so-called 1-2-Conjecture, whose concern is the case of $r = 1$. Within this paper we combine probabilistic approach with purely combinatorial one in order to prove that $\mathrm{ts}_r(G) \leq (2 + o(1))\Delta^{r-1}$ for every integer $r \geq 2$ and each graph G, thus improving the previously best result: $\mathrm{ts}_r(G) \leq 3\Delta^{r-1}$.

1. Introduction

The cornerstone of the field of vertex distinguishing graph colourings is the graph invariant called *irregularity strength*. For a graph $G = (V, E)$ it is usually denoted by $s(G)$ and can be defined as the least integer k so that we may construct an irregular multigraph, i.e. a multigraph with pairwise distinct degrees of all vertices, of G by replacing each edge of G with at most k copies, see [8]. This study thus originated from the basic fact that no graph G with more than one vertex is irregular itself and related research on possible alternative definitions of an irregular graph, see e.g. [7]. Equivalently, $s(G)$ is also defined as the least k so that there exists an edge colouring $c : E \to \{1, 2, \dots, k\}$ such that for every pair $u, v \in E$, $u \neq v$, the sum of colours incident with u is distinct from the sum of colours incident with v. Note that $s(G)$ exists only for graphs without isolated edges and with at most one isolated vertex. It is known that $s(G) \leq n - 1$, where $n = |V|$, for all such graphs, except for K_3, see [1,21]. This tight upper bound can however be improved in the case of graphs with minimum degree $\delta \geq 1$ to $s(G) \leq 6\lceil \frac{n}{\delta} \rceil$ (that yields a better result whenever $\delta > 12$ and for $\delta \in [7, 12]$ if n is larger than a small constant depending on δ), see [16], and to $s(G) \leq (4 + o(1))\frac{n}{\delta} + 4$ for graphs with $\delta \geq n^{0.5} \ln n$, see [19]. Many interesting results, concepts and open problems concerning this graph invariant can also be found e.g. in [6,9–13,16,18,23], and many others.

In [5], Bača et al. introduced a total version of the concept above. Given any graph $G = (V, E)$ and a (not necessarily proper) total colouring $c : V \cup E \to \{1, 2, \dots, k\}$, let

$$w_c(v) := c(v) + \sum_{u \in N(v)} c(uv) \tag{1}$$

denote the *weight* of any vertex $v \in V$, which shall also be called the *sum at* v and denoted simply by $w(v)$ in cases when c is unambiguous from context. The least k for which there exists such colouring with $w(u) \neq w(v)$ for every $u, v \in V$, $u \neq v$, is called the *total vertex irregularity strength* of G and denoted by $\mathrm{tvs}(G)$. In [5] it was proved, among others, that for every

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graph G with n vertices, $\lceil\frac{n+\delta}{\Delta+1}\rceil \leq \text{tvs}(G) \leq n+\Delta-2\delta+1$. Until now the best upper bounds (for graphs with $\delta > 3$) assert that $\text{tvs}(G) \leq 3\lceil\frac{n}{\delta}\rceil+1$, see [3], and $\text{tvs}(G) \leq (2+o(1))\frac{n}{\delta}+4$ for $\delta \geq n^{0.5}\ln n$, see [20]. Many other results e.g. for particular graph families can also be found in [4,22,24,29] and other papers.

In this article we consider a distant generalization of $\text{tvs}(G)$ from [25], motivated among others by the study on distant chromatic numbers, see e.g. [17] for a survey concerning these. For any positive integer r, two distinct vertices at distance at most r in G shall be called *r-neighbours*. We denote by $N^r(v)$ the set of all r-neighbours of $v \in V$ in G, and set $d^r(v) = |N^r(v)|$. The least integer k for which there exists a total colouring $c : V \cup E \to \{1, 2, \ldots, k\}$ such that there are no r-neighbours u, v in G which *are in conflict*, i.e. with $w(u) = w(v)$ (cf. (1)), we call the *r-distant total irregularity strength* of G, and denote by $\text{ts}_r(G)$. It is known that $\text{ts}_r(G) \leq 3\Delta^{r-1}$ for every graph G, see [25], also for a comment implying that a general upper bound for $\text{ts}_r(G)$ cannot be (much) smaller than Δ^{r-1}. In this paper we combine the probabilistic method with algorithmic approach similar to those in e.g. [3,15,20,25] to prove that in fact $\text{ts}_r(G) \leq (2+o(1))\Delta^{r-1}$ (for $r \geq 2$).

Theorem 1. *For every integer $r \geq 2$ there exists a constant Δ_0 such that for each graph G with maximum degree $\Delta \geq \Delta_0$,*

$$\text{ts}_r(G) \leq 2\Delta^{r-1} + 3\Delta^{r-\frac{4}{3}}\ln^2\Delta + 4, \tag{2}$$

hence

$$\text{ts}_r(G) \leq (2+o(1))\Delta^{r-1}$$

for all graphs.

It is also worth mentioning that the case of $r = 1$ was introduced and considered separately in [27], where the well known 1–2-Conjecture concerning this invariant was introduced. It is known that $\text{ts}_1(G) \leq 3$ for all graphs, see Theorem 2.8 in [15], even in case of a natural list generalization of the problem, see [31], though it is believed that the upper bound of 2 should be the optimal general upper bound in both cases, see [27,28,30].

A general upper bound for an analogous graph invariant to $\text{ts}_r(G)$ but defined for edge colourings can be found in [25]. This was improved in [26] by means of a probabilistic approach of a similar flavour as the one presented in this paper.

2. Probabilistic tools

We shall use probabilistic approach in the first part of the proof of Theorem 1, based on the Lovász Local Lemma, see e.g. [2], combined with the Chernoff Bound, see e.g. [14] (Th. 2.1, page 26). We recall these below.

Theorem 2 (*The Local Lemma*). *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most D, and that $\mathbf{Pr}(A_i) \leq p$ for all $1 \leq i \leq n$. If*

$$ep(D+1) \leq 1,$$

then $\mathbf{Pr}\left(\bigcap_{i=1}^{n}\overline{A_i}\right) > 0.$

Theorem 3 (*Chernoff Bound*). *For any $0 \leq t \leq np$,*

$$\mathbf{Pr}(\text{BIN}(n,p) > np+t) < e^{-\frac{t^2}{3np}} \text{ and } \mathbf{Pr}(\text{BIN}(n,p) < np-t) < e^{-\frac{t^2}{2np}} \leq e^{-\frac{t^2}{3np}}$$

where $\text{BIN}(n,p)$ is the sum of n independent Bernoulli variables, each equal to 1 with probability p and 0 otherwise.

Note that if X is a random variable with binomial distribution $\text{BIN}(n,p)$ where $n \leq k$, then we may still apply the Chernoff Bound above, even if we do not know the exact value of n, to prove that $\mathbf{Pr}(X > kp+t) < e^{-\frac{t^2}{3kp}}$ (for $t \leq \lfloor k\rfloor p$).

3. Proof of Theorem 1

Fix an integer $r \geq 2$. Within our proof we shall not specify Δ_0. Instead, we shall assume that $G = (V, E)$ is a graph with sufficiently large maximum degree Δ, i.e. large enough so that all inequalities below are fulfilled.

We first partition V into a subset of vertices with relatively small degrees and a subset of those with large degrees:

$$S = \left\{u \in V : d(u) \leq \Delta^{\frac{2}{3}}\right\};$$
$$B = \left\{u \in V : d(u) > \Delta^{\frac{2}{3}}\right\}.$$

Moreover, for every $v \in V$, we denote $S(v) = N(v) \cap S$, $s(v) = |S(v)|$, $B(v) = N(v) \cap B$, $b(v) = |B(v)|$.

Now we randomly order the vertices of V into a sequence. For this goal, associate with every vertex $v \in V$ a random variable $X_v \sim U[0, 1]$ having the uniform distribution on $[0, 1]$ where all these random variables X_v, $v \in V$ are mutually independent, or in other words pick a (real) number uniformly at random from the interval $[0, 1]$ and associate it with v for

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