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Note

A note on two geometric paths with few crossings for points labeled by integers in the plane

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ABSTRACT

Let S be a set of n points in the plane in general position such that the integers $1, 2, \ldots, n$ are assigned to the points bijectively. Set h be an integer with $1 \leq h < n(n+1)/2$. In this paper we consider the problem of finding two vertex-disjoint simple geometric paths consisting of all points of S such that the sum of labels of the points in one path is equal to h and the paths have as few crossings as possible. We prove that there exists such a pair of paths with at most two crossings between them.

1. Introduction

A *geometric graph* is a graph in the plane whose vertices are points *in general position* (i.e., no three points are colinear), and its edges are straight-line segments. There are several combinatorial and computational properties for simple geometric graphs and much work has been done for them in these decades (see, e.g. [7]). A geometric graph G is said to be *simple* if no two edges of G intersect each other except at their endvertices. In this paper, we focus on simple geometric paths. We can easily see that for any set of points in the plane there exists a simple geometric Hamiltonian path. So paths consisting of all points with some restrictions on points would be interesting. Several results are known for a set of points in which each point is colored either red or blue. See the survey [4] for details. Tokunaga considered two geometric paths each of which consists of all points with the same color for bicolored point sets. For a planar point set S, a geometric path P is an *S-path* if it is simple and $V(P) = S$.

Theorem 1 (*Tokunaga [8]*). *Let $S = R \dot{\cup} B$, where R and B are the sets of red and blue points, respectively. Then S admits an R-path P_R and a B-path P_B such that each edge of P_R crosses at most one edge of P_B and vice versa. Consequently, the number of crossings between P_R and P_B is at most $\min\{|R|, |B|\} - 1$.*

The number of crossings in Theorem 1 is the best possible upper bound in terms of $|R|$ and $|B|$. Put the points of $S = R \dot{\cup} B$ in circular position in the plane such that no two points in R are adjacent in that circle, where $|R| \leq |B|$. It is

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easy to see that the number of crossings between an R-path and a B-path is at least $|R| - 1$. In fact, [8] deals with two monochromatic spanning trees for bicolored point sets and the minimum crossing number between them is determined. Following Tokunaga's results, many researchers have obtained results on finding geometric monochromatic subgraphs for bicolored or multicolored point sets with few crossings. For example, Aichholzer et al. [1] proposed a simpler algorithm to construct two monochromatic spanning trees for bicolored point sets with few crossings. Kano et al. [5] considered spanning trees and cycles for multicolored point sets. In [6], Merino et al. focused on perfect matchings for multicolored point sets and minimum weight perfect matchings with respect to edge lengths.

The purpose of this paper is to reduce the number of crossings by imposing a restriction weaker than bicoloring of the point set as follows. We consider n points in the plane labeled $1, 2, \dots, n$ and an integer h, and we partition the set of points into two sets so that the sum of labels of either set is equal to h. In this case, you have some freedom to select a partition, unlike in the case of bicoloring. For example, if $n = 5$ and $h = 7$, we can choose three partitions $\{\{1, 2, 4\}, \{3, 5\}\}$, $\{\{2, 5\}, \{1, 3, 4\}\}$ and $\{\{3, 4\}, \{1, 2, 5\}\}$. Hence, when we consider the above partition of n points in the plane into two paths, we expect to reduce the number of crossings of the paths. In this paper, we prove the following theorem.

Theorem 2. *Let S be a set of n points in the plane in general position and let the integers $1, 2, \dots, n$ be assigned to the points bijectively. Let h be an integer with $1 \le h < n(n+1)/2$. Then S admits two vertex-disjoint simple geometric paths P_1 and P_2 consisting of all points of S such that*

(1) *the sum of weights of the points in P_1 is equal to h, and*
(2) *the number of crossings between P_1 and P_2 is at most 2.*

Geometric graphs for points labeled bijectively with $\{0, 1, \dots, n-1\}$ are well studied, too. A graph G is *harmonic* if no two edges of G have the same weight, where the weight of each edge is defined the sum of the labels of its endvertices modulo n. In [3], Balogh et al. proposed some algorithms to find large noncrossing harmonic matchings or paths for points in the plane in convex position. Araujo et al. [2] considered simple separable paths for points in convex position. A path P is *separable* if there is a line that crosses all edges of P.

In the next section we will show several results which are used to prove Theorem 2.

2. Preliminaries

We show the following theorem, which plays an important role in the proof of Theorem 2.

Theorem 3. *Let $a_1, a_2, \dots, a_n$ be a permutation on the set $\{1, 2, \dots, n\}$. For any integer h with $1 \le h < n(n+1)/2$, one of the following holds.*

(i) *there exists an integer k such that $\sum_{i=1}^{k} a_i = h$,*
(ii) *there exist integers j, k and l with $j \le k < l$ such that $\sum_{i=1}^{k} a_i - a_j + a_l = h$.*

This theorem says that for any sequence on $\{1, 2, \dots, n\}$ and any integer h $(1 \le h < n(n+1)/2)$, by exchanging at most one pair of elements, we can make a sequence $(a_1, a_2, \dots, a_n)$ on $\{1, 2, \dots, n\}$ such that $\sum_{i=1}^{k} a_i = h$ for some k.

Proof. Set $A_m := \{a_1, a_2, \dots, a_m\}$ for any m, and $\|A_m\| := \sum_{i=1}^{m} a_i$. Suppose that (i) does not hold. Then, there exists an integer t such that $\|A_{t-1}\| < h < \|A_t\|$. Set

$$p = h - \|A_{t-1}\| \text{ and } q = \|A_t\| - h.$$

Then, we have $a_t = p + q$, $p \ge 1$ and $q \ge 1$. For any positive integer i, let b_i be the integer such that $1 \le b_i \le a_t$ and $b_i \equiv -iq \pmod{a_t}$. It is easy to see that for any positive i either $b_{i+1} - b_i = -q$ or $b_{i+1} - b_i = p$ holds and $b_1 = p$. If $b_1 \notin A_t$, then

$$\|A_t\| - a_t + b_1 = (h + q) - (p + q) + p = h.$$

If we set $k = t$, $a_j = a_t$ and $a_l = b_1$, then (ii) holds. So we may assume that $b_1 \in A_{t-1}$ since $b_1 \ne a_t$. On the other hand, we have $b_{a_t} \notin A_{t-1}$ since $b_{a_t} = a_t$. Therefore, there exists an integer r $(1 \le r < a_t)$ such that $b_r \in A_{t-1}$ and $b_{r+1} \notin A_{t-1}$. If $b_{r+1} - b_r = -q$, then

$$\|A_t\| - b_r + b_{r+1} = (h + q) - q = h.$$

If we take $k = t$, $a_j = b_r$ and $a_l = b_{r+1}$, (ii) holds. Otherwise, by $b_{r+1} - b_r = p$,

$$\|A_{t-1}\| - b_r + b_{r+1} = (h - p) + p = h.$$

Set $k = t - 1$, $a_j = b_r$ and $a_l = b_{r+1}$. We have (ii), too. ∎

After some preparations we are ready to prove Theorem 2. Let S be a set of n points in the plane. We denote the convex hull of S by $\mathrm{Conv}(S)$. Let $D(S)$ be the set of points on the boundary of $\mathrm{Conv}(S)$. The following fact is easy.

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