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Graph coloring and Graham's greatest common divisor problem ☆

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ABSTRACT

In this paper we introduce and study two graph coloring problems and relate them to some deep number-theoretic problems. For a fixed positive integer k consider a graph B_k whose vertex set is the set of all positive integers with two vertices a, b joined by an edge whenever the two numbers $a/\gcd(a, b)$ and $b/\gcd(a, b)$ are both at most k. We conjecture that the chromatic number of every such graph B_k is equal to k. This would generalize the greatest common divisor problem of Graham from 1970; in graph-theoretic terminology it states that the clique number of B_k is equal to k. Our conjecture is connected to integer lattice tilings and partial Latin squares completions.

1. Introduction

Let k be a positive integer. Consider the infinite graph B_k on the set $\mathbb{N}$ of all positive integers, in which two vertices a and b are joined by an edge if and only if we have $\frac{a}{b} = \frac{i}{j}$ for some $i, j \in \{1, 2, \dots, k\}$. This graph describes a kind of arithmetic proximity: two numbers are close if they differ only by bounded factors. That is, a is adjacent to b if and only if $\frac{a}{\gcd(a,b)} \leqslant k$ and $\frac{b}{\gcd(a,b)} \leqslant k$.

What is the chromatic number of B_k? Clearly the numbers $1, \dots, k$ form a clique, so there is no hope for coloring that would use less than k colors. Initial inspection of the structure of B_k for small k does not reveal any apparent property that would force us to use more colors and suggests the following conjecture.

Conjecture 1. *Every graph B_k satisfies $\chi(B_k) = k$.*

The problem is closely related to a deep number-theoretic result: Graham's greatest divisor problem. In 1970 Graham conjectured that in any set of n distinct positive integers $a_1, a_2, \dots, a_n$ there is at least one pair satisfying $\frac{a_i}{\gcd(a_i,a_j)} \geqslant n$ [7]. It is equivalent to the assertion that the clique number of B_{n-1} is $n-1$, which is an immediate consequence of Conjecture 1.

It took 26 years to fully confirm Graham's conjecture. Szegedy [11] and Zaharescu [12] gave partial solutions that work for sufficiently large n, but it was not until 1996 when Balasubramanian and Soundararajan [2] obtained a general solution,

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using deep analytic methods. This history suggests that Conjecture 1 may be hard to confirm, but its positive finale also gives some supporting evidence.

In Sections 2 and 3 we describe an intriguing graph coloring model and show how Graham's problem and Conjecture 1 naturally arise in its context. Section 4 develops a geometric interpretation of the problem and relates it to tilings of multidimensional space. Finally, in Section 5 we relate our problem to the problems of existence of certain groups and Latin squares.

2. Fictional graph coloring

In additive type colorings, the color of graphs the color of each vertex v is determined by weights (assigned to vertices or edges) that can be "seen" by v. Typically we would like to obtain a proper coloring using as few different weights as possible. For example, Karoński, Łuczak and Thomason [9] consider a variant where each edge of a graph is assigned a weight from the set $\{1, \dots, k\}$ and the color of a vertex v is a sum of weights assigned to edges incident to v. They conjecture that for $k = 3$ we can always find a weighting that produces a proper coloring (and the best known result is that $k = 5$ suffices [8]). In another variant, we assign weights to vertices and color a vertex v by the sum of weights in the open neighborhood of v (here we need more than a constant number of weights; see [4] and [3]).

In this paper we consider a variation of the latter model: we also weight vertices, but at each edge we insert a mirror, such that each vertex sees a number of copies of its own weight.

Let f be a weighting of the vertices of a graph G by positive integers. Let $\deg(v)$ denote the degree of a vertex v in G. The weighting f is called a *fictional coloring* of G if $f(u)\deg(u) \neq f(v)\deg(v)$ for every pair of adjacent vertices u and v. The least integer k for which there exists a fictional coloring of G, using $\{1, 2, \dots, k\}$ as the set of weights, is called the *fictional chromatic number* of G, and is denoted by $\phi(G)$.

Note that the problem of finding a fictional coloring of a graph is in fact a special case of coloring from lists, where a list of possible colors of vertex v is equal to $\{\deg(v), 2\deg(v), \dots, k\deg(v)\}$. Therefore, $\phi(G)$ is bounded by the list chromatic number of G for every graph G.

Can we improve this bound? Clearly, every regular graph G satisfies $\phi(G) = \chi(G)$. It may seem that for irregular graphs finding a fictional coloring is only easier because there are less possible conflicts of colors. Although such intuition fails in case of list coloring and is nothing near convincing, there is some evidence supporting such a claim and allowing to formulate the following conjecture.

Conjecture 2. *Every graph G satisfies $\phi(G) \leqslant \chi(G)$.*

Conjecture 2 is true for bipartite graphs. To see this, for each vertex v consider the biggest number $e(v)$ such that $2^{e(v)}$ divides $\deg(v)$. We assign weight 2 to vertex v if and only if $e(v)$ is even and v is in the first partite set, or $e(v)$ is odd and v is in the second partite set. It is easily seen that constructed weighting is a fictional coloring.

A slight development of this idea allows us to confirm Conjecture 2 for graphs of chromatic number $p - 1$, where p is a prime.

Theorem 3. *Let p be a prime number and let G be a graph such that $\chi(G) \leqslant p - 1$. Then $\phi(G) \leqslant p - 1$.*

Proof. Fix a proper coloring c of G by non-zero elements of the field $\mathbb{Z}_p$. Our aim is to find a weighting $f : V(G) \to \{1, 2, \dots, p - 1\}$ satisfying the fictional coloring property. Let u be any vertex of degree $\deg(u)$. Write $\deg(u) = ap^j$, with $\gcd(a, p) = 1$. Let x be the unique element of $\{1, 2, \dots, p - 1\}$ such that $ax \equiv c(u) (\operatorname{mod} p)$. Put $f(u) = x$.

We claim that f satisfies the desired property. Indeed, suppose that $\deg(u)f(u) = \deg(v)f(v)$ for some adjacent vertices u and v. Writing $\deg(v) = bp^l$, with $\gcd(b, p) = 1$, and $y = f(v)$, we get

$$ap^jx = bp^ly.$$

Since both numbers ax and by are not divisible by p, there must be $j = l$ (by the unique factorization property). Hence, $ax = by$, which implies that $c(u) = c(v)$, contradicting our assumption on the coloring c. ■

Using Bertrand's postulate (asserting that for every n there is always a prime between n and $2n$) we immediately get a general upper bound on the fictional chromatic number in terms of chromatic number.

Corollary 4. *$\phi(G) \leqslant 2\chi(G)$ for every graph G.*

Notice that one can easily improve this upper bound by applying some stronger results from number theory. For instance, it is known that there is always a prime between $3n$ and $4n$, hence $\phi(G) \leqslant 4/3\chi(G)$ holds for every graph G. Even better—a result of Baker et al. [1], asserts that there is a prime number between $n - n^{0.525}$ and n for sufficiently large n. It shows that Conjecture 2 holds up to an error term of order $o(\chi(G))$.

Corollary 5. *Every graph G satisfies*

$$\phi(G) \leqslant \chi(G) + o(\chi(G)).$$