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Coverings: Variations on a result of Rogers and on the Epsilon-net theorem of Haussler and Welzl

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ABSTRACT

We consider four problems. Rogers proved that for any convex body K, we can cover $\mathbb{R}^d$ by translates of K of density very roughly $d \ln d$. First, we extend this result by showing that, if we are given a family of positive homothets of K of infinite total volume, then we can find appropriate translation vectors for each given homothet to cover $\mathbb{R}^d$ with the same (or, in certain cases, smaller) density.

Second, we extend Rogers' result to multiple coverings of space by translates of a convex body: we give a non-trivial upper bound on the density of the most economical covering where each point is covered by at least a certain number of translates.

Third, we show that for any sufficiently large n, the sphere $\mathbb{S}^2$ can be covered by n strips of width $20n/\ln n$, where no point is covered too many times.

Finally, we give another proof of the previous result based on a combinatorial observation: an extension of the Epsilon-net Theorem of Haussler and Welzl. We show that for a hypergraph of bounded Vapnik–Chervonenkis dimension, in which each edge is of a certain measure, there is a not-too large transversal set which does not intersect any edge too many times.

1. Introduction

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$. The *density* of $\mathcal{F}$ is defined as

$$\lim_{r\to\infty} \frac{\sum_{F\in\mathcal{F}:F\cap B(0,r)\neq\emptyset} \operatorname{vol}(F)}{\operatorname{vol}(B(0,r))}$$

provided that this limit exists, where $B(0,r)$ denotes the ball of radius r centered at the origin, and $\operatorname{vol}(\cdot)$ is the volume (Lebesgue measure).

Let K be a convex body, that is, a compact, convex set with non-empty interior. We denote its *translative covering density*, that is the supremum of the densities of the coverings of $\mathbb{R}^d$ by translates of K, by $\vartheta(K)$, for properties of this quantity cf. [14]. Our starting point is Rogers' estimate [15]:

$$\vartheta(K) \le d\ln d + d\ln\ln d + 5d, \tag{1}$$

which holds for any convex body K in $\mathbb{R}^d$.

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Our first result is an extension of (1). For a family $\mathcal{F}$ of sets in $\mathbb{R}^d$, we say that $\mathcal{F}$ *permits a translative covering* of a subset A of $\mathbb{R}^d$ with density ϑ, if we can select a translation vector $x_F \in \mathbb{R}^d$ for each member F of $\mathcal{F}$ such that $A \subseteq \bigcup_{F\in\mathcal{F}} x_F + F$, and the density of this covering is ϑ.

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^d$, and let $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \ldots\}$, with $0 < \lambda_i$ for all i, be a family of its homothets satisfying*

$$\sum_{i=1}^{\infty} \lambda_i^d = \infty.$$

Let $\Lambda := \{\lambda_1, \lambda_2, \ldots\}$.

(a) *If Λ is bounded, and has a limit point other than zero, then $\mathcal{F}$ permits a covering of space of density $\vartheta(K)$.*
(b) *If Λ is bounded, and has no limit point other than zero, then $\mathcal{F}$ permits a covering of space of density one.*
(c) *If Λ is unbounded, then $\mathcal{F}$ permits a covering of space with maximum multiplicity $4d+1$ (that is, where no point is covered by more than $4d+1$ sets).*

In case (c), we prove maximum multiplicity $2d+1$ in a special case which includes all smooth bodies, see Theorem 2.5. The proofs are in Section 2.

In the proof of Theorem 1.1, we will use a result on covering K by homothets of K.

Theorem 1.2. *Let $K \subseteq \mathbb{R}^d$ be a convex body of volume one, and let $\mathcal{F}$ be a family of positive homothets of K with total volume at least*

$$\begin{cases} (d^3 \cdot \ln d \cdot \vartheta(K) + e)2^d, & \text{if } K = -K, \\ d^3 \cdot \ln d \cdot \vartheta(K) \cdot \binom{2d}{d} + e \cdot 4^d, & \text{in general.} \end{cases}$$

Then $\mathcal{F}$ permits a translative covering of K.

This result is a strengthening of a result of [12], which, in turn is a strengthening of a result of Januszewski [7]. We prove it in Section 2.1. We learned that a stronger bound was recently obtained by Livshyts and Tikhomirov [8] in the case when the dimension d is large. We note that when we apply Theorem 1.2, we will not make use of the actual bound, any bound that depends on the dimension only would suffice.

As a corollary, we will show the following measure theoretic statement, which we will use in the proof of Theorem 1.1.

Corollary 1.3. *Let K be a convex body in $\mathbb{R}^d$, let $A \subset \mathbb{R}^d$ be a set of zero Lebesgue measure. Then any family $\mathcal{F}$ of positive homothets of K with infinite total volume satisfying*

$$\limsup\{\mathrm{vol}\,(F) \;:\; F \in \mathcal{F}\} = 0$$

permits a translative covering of A with density zero.

Our second topic is multiple coverings of space. We denote the infimum of the densities of k-fold coverings of $\mathbb{R}^d$ by translates of K by $\vartheta^{(k)}(K)$. Apart from the estimate that follows from (1) using the obvious fact $\vartheta^{(k)}(K) \leq k\vartheta(K)$, no general estimate has been known. For the Euclidean ball $\mathbf{B}_2^d$ in $\mathbb{R}^d$, G. Fejes Tóth [2,3] gave the non-trivial lower bound $\vartheta^{(k)}(\mathbf{B}_2^d) > c_d k$ for some $c_d > 1$, see more in the survey [4]. We prove

Theorem 1.4. *Let $K \subseteq \mathbb{R}^d$ be a convex body and $k \leq d(2.5\ln d + \ln\ln d)$. Then*

$$\vartheta^{(k)}(K) \leq 20d(2.5\ln d + \ln\ln d).$$

This shows that G. Fejes Tóth's bound (up to a constant factor) is sharp if $k = d\ln d$.

To prove Theorem 1.4, we present in Section 3 a more general statement, Theorem 3.3, which extends [1, Theorem 1.6] and [13, Theorem 1.2].

Our third topic is covering the sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 \;:\; |x| = 1\}$ by strips. For a given point $x \in \mathbb{S}^2$, and $0 \leq w \leq 1$, we call $\{v \in \mathbb{S}^2 \;:\; |\langle v, x\rangle| \leq w\}$ the *strip* centered at x, of Euclidean half-width w.

Theorem 1.5. *For any sufficiently large integer N, there is a covering of $\mathbb{S}^2$ by N strips of Euclidean half-width $\frac{10\ln N}{N}$, with no point covered more than $c\ln N$ times, where c is a universal constant.*

Our study of this question was motivated by a problem at the 2015 Miklós Schweitzer competition posed by András Bezdek, Ferenc Fodor, Viktor Vígh and Tamás Zarnócz on covering the two-dimensional sphere by strips of a given width such that no point is covered too many times.

We note the following dual version of Theorem 1.5, and leave it to the reader to convince themselves that the two versions are equivalent: *For any sufficiently large integer N, we can select N points of $\mathbb{S}^2$ such that each strip of Euclidean half-width $\frac{10\ln N}{N}$ contains at least one and at most $c\ln N$ points, where c is a universal constant.*