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Note

Combinatorial and probabilistic formulae for divided symmetrization

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ABSTRACT

Divided symmetrization of a function $f(x_1, \ldots, x_n)$ is symmetrization of the ratio

$$DS_G(f) = \frac{f(x_1, \ldots, x_n)}{\prod(x_i - x_j)},$$

where the product is taken over the set of edges of some graph G. We concentrate on the case when G is a tree and f is a polynomial of degree $n-1$, in this case $DS_G(f)$ is a constant function. We give a combinatorial interpretation of the divided symmetrization of monomials for general trees and probabilistic game interpretation for a tree which is a path. In particular, this implies a result by Postnikov originally proved by computing volumes of special polytopes, and suggests its generalization.

1. Introduction

Let V be a set of variables, $|V| = m$, say, $V = \{x_1, \ldots, x_m\}$ (but further, we need and allow sets such as $\{x_2, x_3, x_9\}$). It is convenient to think that V is well ordered: $x_1 < x_2 < \cdots < x_m$. For a rational function φ, with coefficients in some field, of variables from V, define its symmetrization as

$$\operatorname{Sym} \varphi = \sum_{\pi} \varphi(\pi_1, \ldots, \pi_m),$$

where summation is taken over all $m!$ permutations π of the variables.

Let f be polynomial of degree d in the variables from V. Then, its divided symmetrization

$$DS(f) := \operatorname{Sym}\left(\frac{f}{\prod_{x,y \in V, x<y}(x-y)}\right)$$

is also polynomial of degree not exceeding $d - m(m-1)/2$. In particular, it vanishes identically when $d < m(m-1)/2$. The reason why $DS(f)$ is a polynomial is the following. Fix variables x, y and partition all summands into pairs corresponding to permutations $(\pi, \sigma\pi)$, where σ is a transposition of x and y. We see that in the sum of any pair, the multiple $x - y$ in the denominator gets cancelled. Thus every multiple is cancelled and so we get polynomial. The symmetrization operators have applications, for instance, in the theory of symmetric functions, see Chapter 7 of the A. Lascoux's book [2].

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Let $G(V, E)$ be a graph on the set of vertices V. We view E as a set of pairs $(x, y) \in V^2, x < y$. We may consider *partial symmetrization in* G, that is,

$$DS_G(f) = \mathrm{Sym}\left(\frac{f}{\prod_{(x,y)\in E}(x-y)}\right).$$

Of course this is a polynomial again of degree at most $d - |E|$ due to the obvious formula

$$DS_G(f) = DS\left(f \cdot \prod_{x<y,(x,y)\notin E}(x-y)\right).$$

If we restrict DS_G to polynomials of degree at most $|E|$, we get a linear functional. The kernel K_G of this functional is particularly structured. First of all, all polynomials of degree less then d lie in K_G. Next, if f has a symmetric factor, i.e., $f = gh$, where g is symmetric and non-constant, then $f \in K_G$. This is true because of the formula $DS_G(gh) = gDS_G(h)$, and the second multiple being equal to 0 since $\deg h < |E|$.

Assume that G is disconnected. That is, $V = U \sqcup W$, and there are no edges of G between U and W: $E = EU \sqcup EW$, where EU, EW are sets of edges joining vertices of U, W respectively. Denote the corresponding subgraphs of G by $GU = (U, EU)$ and $GW = (W, EW)$. Note that both U, W are well ordered sets of variables and thus the above definitions still apply to the subgraphs GU, GW.

Any polynomial f may be represented as a sum $\sum u_i w_i$, where the polynomials u_i depend only on variables from U, while w_i depends only on variables from W (and, of course, the degree $\deg u_i + \deg w_i$ of each summand does not exceed $\deg f$). Assume that $\deg f \leqslant |E|$. Then

$$DS_G(f) = \binom{m}{|U|}\sum_i DS_{GU}(u_i)\cdot DS_{GW}(w_i) \tag{1}$$

(the binomial factor comes from fixing the sets of variables $\pi(U)$ and $\pi(V)$. If $\deg u_i < |EU|$ then the symmetrization $DS_{GU}(u_i)$ is just 0, analogously if $\deg w_i < |EW|$. If $\deg u_i = |EU|$, $\deg w_i = |EW|$, then both $DS_{GU}(u_i)$, $DS_{GW}(w_i)$ are constants and therefore do not depend on the sets of variables $\pi(U)$, $\pi(V)$). It follows that $f \in K_G$ if for any i either $u_i \in K_{GU}$ or $w_i \in K_{GW}$. As already noted above, it is so unless $\deg u_i = |EU|$, $\deg w_i = |EW|$. If f has a factor symmetric in the variables from U, then $DS_{GU}(u_i) = 0$.

Next observation. If $E' \subset E$ and $f = h \cdot \prod_{(x,y)\in E'}(x-y)$ then $DS_G(f) = DS_{G\setminus E'}(h)$. Combining this with our previous argument, we get the following lemma.

Lemma 1. *If $E' \subset E$ and $U \subset V$ is a connected component in $G \setminus E'$, f is divisible by $h\prod_{(x,y)\in E'}(x-y)$, where h is symmetric in variables from U, then $f \in K_G$.*

Denoting by I_G the set of polynomials v such that $vh \in K_G$ provided that $\deg vh \leqslant |E|$ (it is sort of an ideal, but the set of polynomials with restricted degree is not a ring), we have found some elements in I_G: all symmetric polynomials and all polynomials like those in Lemma 1.

Next, we consider the case of partial divided symmetrization w.r.t. tree G on n vertices of a polynomial f, $\deg f = n - 1$. This is a linear functional and we give combinatorial formulae for its values in a natural monomial base.

2. Tree

Definition 1. Let $T = (V, E)$ be a tree on a well ordered set V, $|V| = n$. Let $C := \prod_{x\in V} x^{w(x)+1}$ be a monomial of degree $n - 1$, where we call $w(x) \in \{-1, 0, 1, 2, \ldots\}$ *a weight* of a vertex x. The total weight of all vertices equals -1. For each edge $e = (x, y) \in E$, $x < y$, consider two connected components of the graph $T \setminus e$. The total weight is negative for exactly one of them. If this component contains y, call edge e *regular*, else call it *inversive*. Define sign $\mathrm{sign}(C)$ as $(-1)^{\{\text{number of inversive edges}\}}$. Call a permutation π of the set V to be C-*acceptable* if for all edges $e = (x, y)$, $\pi(x) < \pi(y)$ if and only if e is regular.

Theorem 2. *The partial divided symmetrization $DS_T(C)$ of the monomial C equals the number of C-acceptable permutations times* $\mathrm{sign}(C)$.

Proof. Induction on n. The base case $n = 1$ is obvious. Assume that $n > 1$ and the assertion is valid for $n - 1$. For any monomial C denote by $\tau(C)$ the number of C-acceptable permutations times $\mathrm{sign}(C)$. We need to check that $\tau(C) = DS_T(C)$ for all C. To this end, it suffices to verify the following properties of τ and DS_T:

(i) $\tau(C) - DS_T(C)$ does not depend on C;

(ii) $\sum_{x\in V}\tau(x^{n-1}) = 0 = \sum DS_T(x^{n-1})$.

We start with (i). In turn, it suffices to prove that $\tau(C_1x) - DS_T(C_1x) = \tau(C_1y) - DS_T(C_1y)$, where C_1 is a monomial of degree $n - 2$ and $e = (x, y) \in E$, $x < y$, is an edge of T. We have

$$DS_T(C_1x) - DS_T(C_1y) = DS_T(C_1(x-y)) = DS_{T\setminus e}(C_1).$$