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Note

A property on reinforcing edge-disjoint spanning hypertrees in uniform hypergraphs

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ABSTRACT

Suppose that H is a simple uniform hypergraph satisfying $|E(H)| = k(|V(H)| - 1)$. A k-partition $\pi = (X_1, X_2, \ldots, X_k)$ of $E(H)$ such that $|X_i| = |V(H)| - 1$ for $1 \leq i \leq k$ is a uniform k-partition. Let $P_k(H)$ be the collection of all uniform k-partitions of $E(H)$ and define $\varepsilon(\pi) = \sum_{i=1}^{k} c(H(X_i)) - k$, where $c(H)$ denotes the number of maximal partition-connected sub-hypergraphs of H. Let $\varepsilon(H) = \min_{\pi \in P_k(H)} \varepsilon(\pi)$. Then $\varepsilon(H) \geq 0$ with equality holds if and only if H is a union of k edge-disjoint spanning hypertrees. The parameter $\varepsilon(H)$ is used to measure how close H is being from a union of k edge-disjoint spanning hypertrees.

We prove that if H is a simple uniform hypergraph with $|E(H)| = k(|V(H)| - 1)$ and $\varepsilon(H) > 0$, then there exist $e \in E(H)$ and $e' \in E(H^c)$ such that $\varepsilon(H - e + e') < \varepsilon(H)$. This generalizes a former result, which settles a conjecture of Payan. The result iteratively defines a finite ε-decreasing sequence of uniform hypergraphs $H_0, H_1, H_2, \ldots, H_m$ such that $H_0 = H$, H_m is the union of k edge-disjoint spanning hypertrees, and such that two consecutive hypergraphs in the sequence differ by exactly one hyperedge.

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1. Introduction

We consider finite graphs and finite hypergraphs. Definitions will be introduced in Section 2. Throughout the paper, let $k \geq 1$ be an integer, H denotes a hypergraph, $c(H)$ denotes the number of maximal partition-connected sub-hypergraphs of H, and $\omega(H)$ denotes the number of connected components of H. By definition, for a graph G, partition-connectedness is equivalent to connectedness, and so $c(G) = \omega(G)$. For a hypergraph H, as mentioned in [2], partition-connectedness is a stronger property than connectedness and so $\omega(H)$ and $c(H)$ are different in general. For $X \subseteq E(H)$, $H(X)$ denotes the spanning sub-hypergraph of H with edge set X, whereas $H[X]$ denotes the sub-hypergraph of H induced by X. If $H = (V, E)$ is an r-uniform hypergraph, then the **complement** of H, denoted by H^c, is an r-uniform hypergraph with $V(H^c) = V(H)$ and $E(H^c) = V^{[r]} - E(H)$.

In [7], Payan considered the following problem. Let G be a connected simple graph on $n \geq 2$ vertices and $k(n-1)$ edges. Payan introduced an integral function $\varepsilon(G)$ to measure how the graph G is closed to having k edge-disjoint spanning trees in such a way that G has k edge-disjoint spanning tree if and only if $\varepsilon(G) = 0$. Payan asked the question whether it is always possible to make a finite number of edge exchanges between edges in G and edges not in G so that the corresponding values of ε will be strictly decreasing until it becomes zero. Payan [7] conjectured that the problem has an affirmative answer (confirmed in [4]). In this paper, we study the corresponding problem in hypergraphs.

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Suppose that H is a simple uniform hypergraph satisfying $|E(H)| = k(|V(H)| - 1)$. A k-partition $\pi = (X_1, X_2, \ldots, X_k)$ of $E(H)$ such that $|X_i| = |V(H)| - 1$ for $1 \le i \le k$ is called a **uniform** k**-partition**. Let $P_k(H)$ be the collection of all uniform k-partitions of $E(H)$. We define

$$\varepsilon(\pi) = \sum_{i=1}^{k} c(H(X_i)) - k,$$

and

$$\varepsilon(H) = \min_{\pi \in P_k(H)} \varepsilon(\pi).$$

By definition, $\varepsilon(H) \ge 0$. By Corollary 2.6 of [2] or Theorem 2.2(i), $\varepsilon(H) = 0$ if and only if for every $1 \le i \le k$, $H(X_i)$ is a spanning hypertree of H. Thus $\varepsilon(H) = 0$ if and only if H has k edge-disjoint spanning hypertrees.

The following result was conjectured by Payan [7] and proved in [4].

Theorem 1.1 ([4]). *If G is a simple graph with $|E(G)| = k(|V(G)| - 1)$ and $\varepsilon(G) > 0$, then there exist $e \in E(G)$ and $e' \in E(G^c)$ such that $\varepsilon(G - e + e') < \varepsilon(G)$.*

Note that a simple graph is a 2-uniform hypergraph. The main purpose of this note is to extend Theorem 1.1 to all uniform hypergraphs.

Theorem 1.2. *If H is a simple uniform hypergraph with $|E(H)| = k(|V(H)| - 1)$ and $\varepsilon(H) > 0$, then there exist $e \in E(H)$ and $e' \in E(H^c)$ such that $\varepsilon(H - e + e') < \varepsilon(H)$.*

Remark. (1) The parameter $\varepsilon(H)$, first introduced by Payan in [7] for graphs, can be considered as a measurement that how close H is from being an edge-disjoint union of k spanning hypertrees. Theorem 1.2 iteratively defines a finite ε-decreasing sequence of uniform hypergraphs $H_0, H_1, H_2, \ldots, H_m$ such that $H_0 = H$, H_m is the union of k edge-disjoint spanning hypertrees, and such that any two consecutive hypergraphs in the sequence differ by exactly one hyperedge.
(2) This problem is related to connectivity augmentation problems for a network (modeled as a graph or hypergraph). The traditional connectivity augmentation problem is, adding some edges to increase the connectivity (or edge connectivity, partition connectivity, etc.) of a network. Here a kind of "dynamic augmentation" is considered, i.e.,

- The number of edges in the network does not change.
- In each stage, one edge is deleted and another edge is added from outside, where the two edges are called an edge pair.
- In each stage, partition connectivity augmentation happens, which is so-called "dynamic augmentation".

In this paper, the existence of such edge pairs to augment partition connectivity of a uniform hypergraph is confirmed. It is still interesting to design algorithms to locate those edge pairs.

2. Preliminaries

A **hypergraph** H is a pair (V, E) where V is the vertex set of H and E is a collection of not necessarily distinct nonempty subsets of V, called **hyperedges** or simply **edges** of H. A **loop** is a hyperedge that consists of a single vertex. A hypergraph H is **nontrivial** if $E(H) \neq \emptyset$. A hypergraph H is **simple** if for any $e_1, e_2 \in E(H)$, $e_1 \not\subseteq e_2$. For an integer $r > 0$, and a set V, let $V^{[r]}$ denote the family of all r-subsets of V. A simple hypergraph $H = (V, E)$ is r-uniform if $E \subseteq V^{[r]}$. If $H = (V, E)$ is an r-uniform hypergraph, then the **complement** of H, denoted by H^c, is an r-uniform hypergraph with $V(H^c) = V(H)$ and $E(H^c) = V^{[r]} - E(H)$.

If $W \subseteq V(H)$, the hypergraph (W, E_W), where $E_W = \{e \in E(H) : e \subseteq W\}$ is a **sub-hypergraph induced by the vertex subset** W, and is denoted by $H[W]$. If $X \subseteq E(H)$ and $V_X = \cup_{e \in X} e$, then (V_X, X) is defined as **the sub-hypergraph induced by the edge subset** X and is denoted by $H[X]$.

A hypergraph H is **connected** if there is a hyperedge intersecting both W and $V - W$ for every non-empty proper subset W of $V(H)$. A **connected component** of a hypergraph H is a maximal connected sub-hypergraph of H. A hypergraph H is k**-partition-connected** if $\|P\| \ge k(|P| - 1)$ for every partition P of $V(H)$, where $|P|$ denotes the number of classes in P and $\|P\|$ denotes the number of edges intersecting at least two classes of P. Equivalently, H is k-partition-connected if, for any subset $X \subseteq E(H)$, $|X| \ge k(\omega(H - X) - 1)$. A 1-partition-connected hypergraph is also referred as a **partition-connected** hypergraph. It follows from definition that a graph is partition-connected if and only if it is connected. In general, partition-connected hypergraphs must be connected, but a connected hypergraph may not be partition-connected. The **partition connectivity** of H is the maximum k such that H is k-partition-connected.

A hypergraph H is a **hyperforest** if for every nonempty subset $U \subseteq V(H)$, $|E(H[U])| \le |U| - 1$. A hyperforest T is called a **hypertree** if $|E(T)| = |V(T)| - 1$. By a **hypercircuit**, we mean a hypergraph C with $|E(C)| = |V(C)|$ but $|X| < |V(C[X])|$ for any proper subset $X \subset E(C)$. For a hypergraph H, let $\tau(H)$ be the maximum number of edge-disjoint spanning hypertrees in H and $a(H)$ be the minimum number of edge-disjoint hyperforests whose union is $E(H)$. For a graph G, $\tau(G)$ is the spanning tree packing number of G and $a(G)$ is the arboricity of G.

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