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Maximality of the signless Laplacian energy

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ABSTRACT

We study the problem of determining the graph with n vertices having largest signless Laplacian energy. We conjecture it is the complete split graph whose independent set has (roughly) $2n/3$ vertices. We show that the conjecture is true for several classes of graphs. In particular, the conjecture holds for the set of all complete split graphs of order n, for trees, for unicyclic and bicyclic graphs. We also give conditions on the number of edges, number of cycles and number of small eigenvalues so the graph satisfies the conjecture.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$, $|V(G)| = n$, and edge set $E(G)$, $|E(G)| = m$. The *Laplacian matrix* and the *signless Laplacian matrix* of G are given by $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$, respectively, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of vertex degrees of G. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ and $q_1 \leq q_2 \leq \cdots \leq q_n$ be the eigenvalues of the matrices $L(G)$ and $L^+(G)$. When more than one graph is under consideration, we write $\mu_i(G)$ and $q_i(G)$ instead of μ_i and q_i.

The *Laplacian energy* of G, [12], is defined as

$$LE(G) = \sum_{i=1}^{n} |\mu_i(G) - \overline{d}| \tag{1}$$

and the *signless Laplacian energy* of G, [11], is defined as

$$LE^+(G) = \sum_{i=1}^{n} |q_i(G) - \overline{d}|, \tag{2}$$

where $\overline{d} = \frac{2m}{n}$ is the average degree of G. More details on Laplacian energy can be found in [4,12,18,19]. The signless Laplacian energy is a recent concept (2010). This is likely one of the reasons why the number of references on this topic is limited. In fact, to our knowledge, most of the work on this may be seen in Refs. [1,5–7,11].

Concerning this relatively new parameter, Abreu et al. [1] and Das and Mojallal [5] established bounds for the signless Laplacian energy of a graph G relating to the Laplacian energy and the energy (sum of the absolute values of the eigenvalues of the adjacency matrix $A(G)$) of G. Abreu et al. also established some bounds for the signless Laplacian energy involving the number of vertices n and the number of edges m. Gutman et al. [11], Das and Mojallal [6] and Das et al. [7] related the signless Laplacian energy of a graph G with the energy of the line graph of G.

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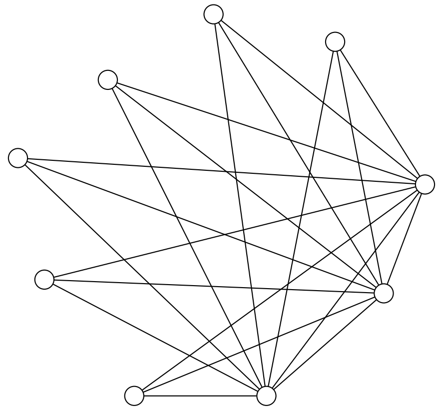

Fig. 1. A complete split graph with $n = 9$ vertices.

A classical problem involving the Laplacian energy is to find extremal graphs involving this parameter, that is to find the graph on n vertices with the largest Laplacian energy. Some of the relevant references on this topic are [2,9,13,17]. Concerning the signless Laplacian energy this is essentially an unheard problem. We are unaware of studies dealing with graphs having largest signless Laplacian energy other than [3], where we partially solved this problem in the class of unicyclic graphs.

Our main goal in this paper is to consolidate this extremal problem for the signless Laplacian energy. We first state the following conjecture concerning the general graph having the largest signless Laplacian energy. Recall that a graph is said to be *split* if its set of vertices can be partitioned into an independent set and a clique. And is said to be *complete split* graph if any vertex of the independent set is adjacent to all vertices of the clique. Fig. 1 illustrates a complete split graph with 9 vertices and independent set with 6 vertices.

Conjecture 1. *For $n \geq 6$, the graph with n vertices having largest signless Laplacian energy is the complete split graph whose independent set has $\left\lceil \frac{2n-1}{3} \right\rceil$ vertices and the clique has $\left\lfloor \frac{n+1}{3} \right\rfloor$ vertices.*

We confirmed this conjecture computationally in the class of threshold graphs for $n \leq 31$ vertices.

Furthermore we prove that the conjecture is true in the class of complete split graphs with n vertices. More precisely, the following result is proven in Section 3.

Theorem 2. *For $n < 6$, the complete split graph with n vertices having largest signless Laplacian energy is the complete graph. For $n \geq 6$, the complete split graph with n vertices having largest signless Laplacian energy is the complete split graph with independent set of $\left\lceil \frac{2n-1}{3} \right\rceil$ vertices and clique of $\left\lfloor \frac{n+1}{3} \right\rfloor$ vertices.*

We determine classes of graphs satisfying restrictions on some spectral parameters whose signless Laplacian energy is smaller than or equal to the signless Laplacian energy of this candidate. In particular, we prove that all trees of order n and all unicyclic and bicyclic graphs of order n have signless Laplacian energy smaller than the candidate graph. We also prove sufficient conditions for a general c-cyclic graph to have smaller signless Laplacian energy than that the candidate.

The paper is organized as follows. Section 2 introduces some notation and preliminary results that will be used in the paper. In Section 3, we compute the signless Laplacian energy of any complete split graph and we will prove Theorem 2. Finally, in Section 4 we give conditions based on parameters like number of edges, number of small eigenvalues and number of cycles so that the graph satisfies Conjecture 1. We also present classes of graphs having signless Laplacian energy smaller than or equal to the signless Laplacian energy of the candidate graph.

2. Preliminaries

Since a complete split graph is a threshold graph, we revise some properties of these graphs that will be used here in this note.

A graph G with n vertices is a *threshold graph* if we can obtain it through an iterative process. The threshold graph with binary sequence $(b_i) = b_1 b_2 \cdots b_n$, is obtained as follows. Beginning with an empty graph, and for each $i = 1, \ldots, n$, one of the following operations is performed

(i) addition of an isolated vertex, if $b_i = 0$;
(ii) addition of a dominating vertex (vertex adjacent to all previous vertices), if $b_i = 1$.