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Nowhere-zero 3-flow of graphs with small independence number

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ABSTRACT

Tutte's 3-flow conjecture states that every 4-edge-connected graph admits a nowhere-zero 3-flow. In this paper, we characterize all graphs with independence number at most 4 that admit a nowhere-zero 3-flow. The characterization of 3-flow verifies Tutte's 3-flow conjecture for graphs with independence number at most 4 and with order at least 21. In addition, we prove that every odd-5-edge-connected graph with independence number at most 3 admits a nowhere-zero 3-flow. To obtain these results, we introduce a new reduction method to handle odd wheels.

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1. Introduction

Graphs in this paper are finite and loopless, but may contain parallel edges. We follow [2] for undefined terms and notation. For a graph G, let $\alpha(G)$, $\kappa'(G)$, and $\delta(G)$ denote the independence number, the edge-connectivity, and the minimum degree of G, respectively. For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) | u \in U, w \in W\}$. When $U = \{u\}$ or $W = \{w\}$, we use $[u, W]_G$ or $[U, w]_G$ for $[U, W]_G$, respectively. We also use $\partial_G(S) = [S, V(G) - S]_G$ to denote an edge-cut of G. The subscript G may be omitted when G is understood from the context.

Let $D = D(G)$ be an orientation of G. For each $v \in V(G)$, let $E^+_D(v)$ ($E^-_D(v)$, respectively) be the set of all arcs directed out from (into, respectively) v. An *integer flow* (D, f) of G is an orientation D and a mapping $f : E(G) \mapsto \mathbb{Z}$ such that, for every vertex $v \in V(G)$,

$$\sum_{e \in E^+_D(v)} f(e) - \sum_{e \in E^-_D(v)} f(e) = 0.$$

An integer flow (D, f) is called a *nowhere-zero k-flow* if $1 \le |f(e)| \le k - 1$, for each edge $e \in E(G)$.

Let $d^+_D(v) = |E^+_D(v)|$ and $d^-_D(v) = |E^-_D(v)|$ denote the out-degree and the in-degree of v under the orientation D, respectively. A graph G admits a *modulo 3-orientation*, or a mod 3-orientation for short if it has an orientation D such that $d^+_D(v) - d^-_D(v) \equiv 0 \pmod 3$ for every vertex $v \in V(G)$. It is well-known that a graph admits a nowhere-zero 3-flow if and only if it admits a modulo 3-orientation (see [7,15,18]). Therefore, in this paper, we will study nowhere-zero 3-flow in terms

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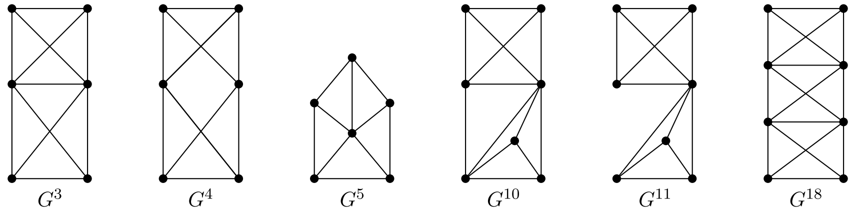

Fig. 1. Graphs in Theorems 1.3 and 1.4.

of modulo 3-orientation. The odd-edge-connectivity of a graph is defined as the minimum size of an edge-cut of odd size. A graph with low edge-connectivity may have high odd-edge-connectivity.

Tutte posed the following famous 3-Flow Conjecture, which appeared in 1970 s (see [2]).

Conjecture 1.1. *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

Thomassen [14] settled the weak version of 3-Flow Conjecture with edge-connectivity 8 replacing 4 and his result was further improved by Lovász, Thomassen, Wu and Zhang [11].

Theorem 1.2 (*Lovász et al. [11]*)**.** *Every odd-7-edge-connected graph admits a nowhere-zero 3-flow.*

Jaeger et al. [8] introduced the concept of group connectivity as generalizations of nowhere-zero flows. Let $Z(G, \mathbb{Z}_3) = \{b : V(G) \to \mathbb{Z}_3 \mid \sum_{v\in V(G)} b(v) \equiv 0 \pmod 3\}$. A graph G is $\mathbb{Z}_3$-*connected* if, for any $b \in Z(G, \mathbb{Z}_3)$, there is an orientation D such that $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod 3$ for every vertex $v \in V(G)$. Let $\langle \mathbb{Z}_3 \rangle$ denote the family of all $\mathbb{Z}_3$-connected graphs. Jaeger et al. [8] pointed out that not every 4-edge-connected graph is $\mathbb{Z}_3$-connected, and they further conjectured that *every 5-edge-connected graph is* $\mathbb{Z}_3$*-connected.* This conjecture, if true, implies Tutte's 3-Flow Conjecture as Kochol [9] showed that the 3-Flow conjecture is equivalent to its restriction to 5-edge-connected graphs.

Luo et al. [12] characterized graphs with independence number two that admit a nowhere-zero 3-flow.

Theorem 1.3 (*Luo et al. [12]*)**.** *Let G be a bridgeless graph with independence number $\alpha(G) \le 2$. Then G admits a nowhere-zero 3-flow if and only if G cannot be contracted to K_4 or G^3, and G is not one of three exceptional graphs, G^3, G^5, G^{18} (see Fig. 1).*

Yang et al. [17] further refined this result to characterize 3-edge-connected $\mathbb{Z}_3$-connected graphs with independence number two. To state their theorem, we need to introduce the concept of $\langle \mathbb{Z}_3 \rangle$-reduction first. Note that a K_1 is $\mathbb{Z}_3$-connected, which is called a trivial $\mathbb{Z}_3$-connected graph, and thus for any graph G, every vertex lies in a maximal $\mathbb{Z}_3$-connected subgraph of G. Let $H_1, H_2, \ldots, H_c$ denote the collection of all maximal $\mathbb{Z}_3$-connected subgraph of G. We call $G' = G/(\cup_{i=1}^c E(H_i))$ the $\langle \mathbb{Z}_3 \rangle$-*reduction* of G, and we say that G is $\langle \mathbb{Z}_3 \rangle$-*reduced to* G'. A graph G is $\langle \mathbb{Z}_3 \rangle$-*reduced* if G does not have a nontrivial $\mathbb{Z}_3$-connected subgraph. By definition, the $\langle \mathbb{Z}_3 \rangle$-reduction of a graph is always $\langle \mathbb{Z}_3 \rangle$-reduced. It is shown in [10] that a graph G admits a nowhere-zero 3-flow (is $\mathbb{Z}_3$-connected, respectively) if and only if its $\langle \mathbb{Z}_3 \rangle$-reduction admits a nowhere-zero 3-flow (is $\mathbb{Z}_3$-connected, respectively). Moreover, the potential minimal counterexamples to Conjecture 1.1 must be $\langle \mathbb{Z}_3 \rangle$-reduced graphs. Therefore in order to describe nowhere-zero 3-flow and $\mathbb{Z}_3$-connectedness properties of certain family of graphs, it is sufficient to characterize all $\langle \mathbb{Z}_3 \rangle$-reductions of this family.

Theorem 1.4 (*Yang et al. [17]*)**.** *Let G be a 3-edge-connected graph with $\alpha(G) \le 2$. If G is not one of the 18 graphs of order at most 8, then G is $\mathbb{Z}_3$-connected if and only if G cannot be $\langle \mathbb{Z}_3 \rangle$-reduced to one of the graphs in $\{K_4, G^3, G^4, G^{10}, G^{11}\}$ (see Fig. 1).*

The purpose of this paper is to further extend Theorem 1.3 to graphs with independence number at most 4, and thus resolve the 3-Flow Conjecture for this family of graphs.

Denote $\mathcal{F}_1 = \{H \mid H$ is $\langle \mathbb{Z}_3 \rangle$-reduced without mod 3-orientation, $2 \le |V(H)| \le 15$, $\alpha(H) \le 4$ and $\kappa'(H) \le 3\}$, and let $\mathcal{F}_2 = \{H \mid H$ has no mod 3-orientation and $14 \le |V(H)| \le 20\}$.

Theorem 1.5. *Let G be a graph with $\alpha(G) \le 4$. Then G admits a nowhere-zero 3-flow if and only if $G \notin \mathcal{F}_2$ and the $\langle \mathbb{Z}_3 \rangle$-reduction of G is not in $\mathcal{F}_1$.*

Since each graph in $\mathcal{F}_1$ is of edge-connectivity at most 3, Theorem 1.5 immediately leads the following, which verifies the 3-Flow Conjecture for graphs with at least 21 vertices and independence number at most 4.

Theorem 1.6. *Every 4-edge-connected graph G with $|V(G)| \ge 21$ and $\alpha(G) \le 4$ admits a nowhere-zero 3-flow.*

In Section 3, we will show that Theorem 1.5 is equivalent to Theorem 1.6 (Lemma 3.3).

For graphs with independence number at most 3, we can eliminate the order requirement in Theorem 1.6 and prove the following theorem.

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