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On the girth and diameter of generalized Johnson graphs

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ABSTRACT

Let $v > k > i$ be non-negative integers. The generalized Johnson graph, $J(v, k, i)$, is the graph whose vertices are the k-subsets of a v-set, where vertices A and B are adjacent whenever $|A \cap B| = i$. In this article, we derive general formulas for the girth and diameter of $J(v, k, i)$. Additionally, we provide a formula for the distance between any two vertices A and B in terms of the cardinality of their intersection.

1. Introduction

Let $v > k > i$ be non-negative integers. The *generalized Johnson graph*, $X = J(v, k, i)$, is the graph whose vertices are the k-subsets of a v-set, where vertices A and B are adjacent whenever $|A \cap B| = i$. Generalized Johnson graphs were introduced by Chen and Lih in [2]. Special cases include the Kneser graphs $J(v, k, 0)$, the odd graphs $J(2k + 1, k, 0)$, and the Johnson graphs $J(v, k, k - 1)$. The Johnson graph $J(v, k, k - 1)$ is well known to have diameter $\min\{k, v - k\}$, and formulas for the distance and diameter of Kneser graphs were proved in [5].

Generalized Johnson graphs have also been studied under the name *uniform subset graphs*, and a result in [3] offers a general formula for the diameter of $J(v, k, i)$. However, that formula gives incorrect values when $i > \frac{2}{3}k$, an important case that includes the Johnson graphs. In this paper we extend (and, in places, correct) those expressions for the diameter of generalized Johnson graphs and we additionally provide a formula for the girth.

Note that it is possible to extend the definition of $X = J(v, k, i)$ to include $v \geq k \geq i$. However, X is an empty graph when $k = i$ or $v = k$. If $v = 2k$ and $i = 0$, then X is isomorphic to the disjoint union of copies of K_2. Furthermore, by taking complements, the graphs $J(v, k, i)$ and $J(v, v-k, v-2k+i)$ are easily seen to be isomorphic (see [4, p. 11]). For the remainder of this article, we will be concerned with generalized Johnson graphs that are connected, so we make the following global definition.

Definition 1.1. Assume $v > k > i$ are nonnegative integers, and let $X = J(v, k, i)$ denote the corresponding generalized Johnson graph. To avoid trivialities, further assume that $v \geq 2k$, and that $(v, k, i) \neq (2k, k, 0)$.

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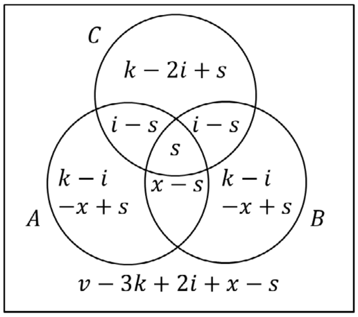

Fig. 1. Diagram for proof of Lemma 2.1.

2. Girth

In this section we derive an expression for the girth $g(X)$ of a generalized Johnson graph, X. We begin with a lemma that characterizes when two vertices have a common neighbor.

Lemma 2.1. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Then A and B have a common neighbor if and only if $x \geq \max\{-v + 3k - 2i, 2i - k\}$.*

Proof. Vertices A and B have a common neighbor C if and only if there exists a nonnegative integer s, such that every region in the above diagram (Fig. 1) has nonnegative size.
By simplifying the resulting inequalities, we find that A and B have a common neighbor if and only if there exists $s \in \mathbb{Z}$, such that

$$\max\{0,\ i + x - k,\ 2i - k\} \leq s \leq \min\{x,\ i,\ v - 3k + 2i + x\}.$$

Such an integer s exists if and only if the expression on the left-hand side above does not exceed the expression on the right-hand side. Under our global assumptions, this is equivalent to $x \geq \max\{-v + 3k - 2i, 2i - k\}$. □

Lemma 2.2. *With reference to Definition 1.1, the girth $g(X) = 3$ if and only if $v \geq 3(k - i)$.*

Proof. The graph X contains a 3-cycle if and only if there exist adjacent vertices A and B that have a common neighbor. By Lemma 2.1, this occurs if and only if $i \geq \max\{-v+3k-2i, 2i-k\}$. Since $i \geq 2i-k$ holds in all $J(v, k, i)$ graphs, this condition is equivalent to $v \geq 3(k - i)$. □

A sufficient condition for the girth to be at most 4 is the existence of a 4-cycle.

Lemma 2.3. *With reference to Definition 1.1, if $(v, k, i) \neq (2k + 1, k, 0)$ then $g(X) \leq 4$.*

Proof. We proceed in two cases.
Case 1: $i \geq 2$ or $v > 2k + 1$. In this case we get that $v \geq 2k - i + 2$. So we can find disjoint sets, A_1, A_2, A_3, A_4, and B_1, B_2, and C such that $|A_1| = |A_2| = |A_3| = |A_4| = 1$, and $|B_1| = |B_2| = k - i - 1$, and $|C| = i$. Then

$$A_1 \cup B_1 \cup C, \quad A_2 \cup B_2 \cup C, \quad A_3 \cup B_1 \cup C, \quad A_4 \cup B_2 \cup C$$

is a 4-cycle in X.
Case 2: $i = 1$. In this case, since $v \geq 2k$, we can find disjoint sets A_1, A_2, A_3, A_4 and B_1, B_2 such that $|A_1| = |A_2| = |A_3| = |A_4| = 1$ and $|B_1| = |B_2| = k - 2$. Then

$$A_1 \cup A_2 \cup B_1, \quad A_2 \cup A_3 \cup B_2, \quad A_3 \cup A_4 \cup B_1, \quad A_4 \cup A_1 \cup B_2$$

is a 4-cycle in X. □

Combining the above lemmas, we obtain a general expression for the girth.

Theorem 2.4. *With reference to Definition 1.1, the girth of X is given by*

$$g(X) = \begin{cases} 3 & \text{if } v \geq 3(k-i); \\ 4 & \text{if } v < 3(k-i) \text{ and } (v,k,i) \neq (2k+1,k,0); \\ 5 & \text{if } (v,k,i) = (5,2,0); \\ 6 & \text{if } (v,k,i) = (2k+1,k,0) \text{ and } k > 2. \end{cases}$$

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