Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

A panconnectivity theorem for bipartite graphs

Hui Du [a], Ralph J. Faudree [b], Jenő Lehel [c,d,*], Kiyoshi Yoshimoto [e]

[a] *Nagakura 2722, Karuizawa, Nagano 389-0111, Japan*
[b] *University of Memphis, Memphis, TN 38152, United States*
[c] *University of Louisville, Louisville, KY 40292, United States*
[d] *Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary*
[e] *Nihon University, Tokyo 101-8308, Japan*

ARTICLE INFO

Article history:
Received 2 June 2017
Accepted 16 August 2017
Available online xxxx

Keywords:
Bipartite graph
Circumference
Panconnectivity

ABSTRACT

Let G be a simple $m \times n$ bipartite graph with $m \geq n$. We prove that if the minimum degree of G satisfies $\delta(G) \geq m/2 + 1$, then G is bipanconnected: for every pair of vertices x, y, and for every appropriate integer $2 \leq \ell \leq 2n$, there is an x, y-path of length ℓ in G.

1. Introduction

It was proved by Williamson in [7] that the minimum degree condition $\delta(G) \geq n/2 + 1$ implies the *panconnectivity* of a simple graph G of order n: for every pair of vertices x, y, and for every integer $2 \leq \ell \leq n - 1$, there is an x, y-path of length ℓ in G. Panconnectivity, related concepts and applications are investigated in [4] and [6]. The case of paths of any odd length between vertices in different partite sets of a balanced bipartite graph was considered by Amar et al. in [1]. In this paper we deal with the Panconnectivity of bipartite graphs in general, and prove Theorem 1 we needed to apply in [3].

Let G be a (simple) bipartite graph with partite sets A and B, $|A| \geq |B|$, and let $x, y \in A \cup B$. Let $\ell \geq 2$ be an integer such that

$\ell \leq 2|B| - 1$ and it is odd, if $x \in A$ and $y \in B$,
$\ell \leq 2|B| - 2$ and it is even, if $x, y \in B$,
$\ell \leq 2|B|$ and it is even, if $x, y \in A$ and $|A| > |B|$.

Then we say that G is *bipanconnected*, if for every $x, y \in V(G)$ and for every appropriate integer ℓ as above, G has an x, y-path of length ℓ.

A bipartite graph with partite sets of cardinality m and n is called an $m \times n$ *bigraph*. Our main result is a bipartite version of Williamson's theorem, and it extends the special case $m = n$ proved by Amar et al. in [1].

Theorem 1. *Let G be an $m \times n$ bigraph with $m \geq n \geq 2$. If $\delta(G) \geq m/2 + 1$, then G is bipanconnected.*

The minimum degree condition in Theorem 1 cannot be relaxed. As an example, let $m = 2t + 1$, $t \geq 1$, and take two copies of a complete $(t + 1) \times (t + 1)$ bigraph sharing a common edge xy. Then $\delta = t + 1 = (m + 1)/2$, and for $\ell = m + 2, m + 4, \ldots, 2m - 1$, the bigraph has no x, y-path of length ℓ.

In our proof of Theorem 1 we use Jackson's theorem on the circumference of a bipartite graph. The maximum cycle length, $c(G)$, of a graph G is called its *circumference*. A connected graph with no cut vertex is called 2-*connected*.

* Correspondence to: 10401 Christina Court, Louisville, KY 40223, United States.
E-mail address: j0lehe01@louisville.edu (J. Lehel).

http://dx.doi.org/10.1016/j.disc.2017.08.024
0012-365X/© 2017 Elsevier B.V. All rights reserved.

Theorem 2 (*Jackson [5]*). *Let G be a 2-connected $m \times n$ bigraph with $m \geq n$, and let s and t be the minimum degree in the partite sets of cardinality m and n, respectively. Then $\frac{1}{2}c(G) \geq \min\{n, s+t-1, 2t-2\}$. Moreover, if $m = n$ and $s = t = \delta(G)$, then $\frac{1}{2}c(G) \geq \min\{m, 2\delta(G)-1\}$.* □

2. Proof of Theorem 1

We adapt basic graph theory notations like in [2]. Let G be a simple graph. The set of all adjacencies of a vertex $v \in V(G)$ in $S \subset V(G)$ is denoted by $N_S(v)$, and we set $d_S(v) = |N_S(v)|$.

If C is a cycle and $v \in C$, then v^+ and v^- will, respectively, denote the successor and the predecessor of v according to a fixed orientation of C. The reverse orientation of C will be denoted by $\overleftarrow{C}$.

For $S \subseteq C$, the set of all successors and predecessors of the vertices in S will be denoted by S^+ and S^-, respectively. The vertex at distance k following $v \in C$ along C, called the kth successor of v, is denoted by v^{+k}. We denote by S^{+k} the set of the kth successors of all vertices in S. Similar notation will be used for the kth predecessors, namely v^{-k} and S^{-k}.

For $x, y \in C$ a subpath of C with end vertices x and y is denoted by (x, C, y) or $(x, \overleftarrow{C}, y)$. According to these notations, if $k < |C|$, then (v, C, v^{+k}), $(v^{+k}, \overleftarrow{C}, v)$ or (v^{-k}, C, v) are all subpaths of length k in C.

Let G be an $m \times n$ bigraph with $m \geq n$ and satisfying $\delta(G) \geq m/2+1$. Given $x, y \in V(G)$ and an appropriate length ℓ, we may assume that $\ell \geq 4$. Indeed, from the minimum degree condition it follows that any two vertices of the same partite set have at least two common neighbors; this implies that there is an x, y-path of length 2 or 3. The theorem is obviously true for complete bigraphs; this is implied by the degree condition, for all $n \leq 4$, except the case $m = n = 4$. In this exceptional case G is a supergraph of the 3-regular 4×4 bigraph, $K_{4,4} - 4K_2$, which is bipanconnected, by inspection. From now on we assume that $m \geq n \geq 5$.

Let A and B be the partite classes of G, where $|A| = m$, $|B| = n$. For a set $S \subseteq V(G)$ we use $S_A = S \cap A$ and $S_B = S \cap B$; similarly, for a subgraph $H \subseteq G$ we use H_A and H_B to denote the partite classes of H.

For $k \geq 1$, a connected graph is k-connected, if the removal of less than k vertices does not disconnect the graph. We need the following simple lemma.

Lemma 3. *For $m \geq n \geq 5$, let G be an $m \times n$ bigraph. If $\delta(G) \geq m/2+1$, then G is 4-connected.*

Proof. Let $S \subset V(G)$ be a cut set of G. Due to the minimum degree condition any two vertices in one partite set have at least two common neighbors. This means that either $A \setminus S$ belongs to one connected component G_1 of $G - S$, or $|S_B| \geq 2$. In the first case the degree of any vertex w of $B \setminus S$ not in G_1 satisfies $|S_A| \geq d(w) \geq \lceil m/2 \rceil + 1 \geq 4$, and $|S| \geq 4$ follows. In the second case we have $|S_B| \geq 2$, and in the same way, by symmetry, we obtain $|S_A| \geq 2$. Thus $|S| \geq 4$ follows. □

Depending on G and the location of x and y there are three cases to consider (see the definition of bipanconnectivity).

Case 1: $x \in A$, $y \in B$, ℓ is odd, and $5 \leq \ell \leq 2n-1$.

By Lemma 3, G is 4-connected. Thus $H = G - \{x, y\}$ is a 2-connected $m' \times n'$ bigraph, where $m' = m-1$, $n' = n-1$. Since $\delta(H) \geq m/2$, by Theorem 2 we obtain

$$\frac{1}{2}c(H) \geq \begin{cases} \min\{n-1, m-1\} = n-1, & \text{for } n = m. \\ \min\{n-1, m-1, m-2\} = n-1, & \text{for } n < m. \end{cases}$$

Therefore, if C is a longest cycle of H, then $C_B = B \setminus \{y\}$, hence $d_C(x) \geq m/2$.

First suppose that y has all neighbors in $C_A \cup \{x\}$. If there is a vertex $v \in N_C(x)^{+(\ell-2)} \cap N_C(y)$, then $(x, v^{-(\ell-2)}, C, v, y)$ is an x, y-path of length ℓ. If $N_C(x)^{+(\ell-2)} \cap N_C(y) = \emptyset$, then using $d_C(y) \geq m/2$ we obtain

$$m \leq d_C(x) + d_C(y) = |N_C(x)^{+(\ell-2)}| + |N_C(y)| \leq |C_A| \leq m-1,$$

a contradiction.

Now suppose that yy' is an edge of $G - C$ for some $y' \neq x$. As before, if $v \in N_C(x)^{+(l-3)} \cap N_C(y') \neq \emptyset$, then $(x, v^{-(\ell-3)}, C, v, y', y)$ is a required x, y-path. Otherwise, since $d_C(y') \geq m/2$, we have

$$m \leq d_C(x) + d_C(y') \leq |N_C(x)^{+(l-3)}| + |N_C(y')| \leq |C_B| \leq m-1,$$

a contradiction.

Case 2: $x, y \in B$, ℓ is even, and $4 \leq \ell \leq 2n-2$.

By Lemma 3, $H = G - \{x, y\}$ is a 2-connected $m' \times n'$ bigraph where $m' = m$, $n' = n-2$. Furthermore, $d_H(v) \geq m/2-1$, for $v \in A$, and $d_H(v) \geq m/2+1$, for $v \in B \setminus \{x, y\}$. By Theorem 2, we obtain

$$\frac{1}{2}c(H) \geq \min\{n-2, m-1, m\} = n-2,$$

hence for a longest cycle C of H we have $C_B = B \setminus \{x, y\}$.