ARTICLE IN PRESS
Discrete Mathematics ■ (■■■■) ■■■–■■■

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Independent domination in subcubic graphs of girth at least six

Gholamreza Abrishami [a], Michael A. Henning [b,*]

[a] Department of Applied Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran
[b] Department of Pure and Applied Mathematics, University of Johannesburg, Auckland Park, 2006 South Africa

ARTICLE INFO

Article history:
Received 12 November 2016
Received in revised form 14 August 2017
Accepted 17 August 2017
Available online xxxx

Keywords:
Independent domination
Cubic graphs

ABSTRACT

A set S of vertices in a graph G is an independent dominating set of G if S is an independent set and every vertex not in S is adjacent to a vertex in S. The independent domination number, $i(G)$, of G is the minimum cardinality of an independent dominating set. In this paper, we extend the work of Henning, Löwenstein, and Rautenbach (2014) who proved that if G is a bipartite, cubic graph of order n and of girth at least 6, then $i(G) \le \frac{4}{11}n$. We show that the bipartite condition can be relaxed, and prove that if G is a cubic graph of order n and of girth at least 6, then $i(G) \le \frac{4}{11}n$.

1. Introduction

An *independent set* of a graph G is a set of vertices of G such that no two vertices in the set are adjacent, while a *dominating set* of G is a set S of vertices of G such that every vertex outside S is adjacent to a vertex in S. An *independent dominating set*, abbreviated ID-*set*, in a graph is a set that is both dominating and independent. Equivalently, an independent dominating set is a maximal independent set. The *independent domination number* of a graph G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set in G. Independent dominating sets have been studied extensively in the literature (see, for example, [1,2,4,5,7,9–13] and the so-called domination book [6]). A recent survey on independent domination in graphs can be found in [3].

The *order* of the graph G with vertex set is $n(G) = |V(G)|$ and the *size* of G with edge set $E(G)$ is $m(G) = |E(G)|$. A *neighbor* of a vertex v in the graph G is a vertex adjacent to v. The *open neighborhood* of v, denoted $N_G(v)$, is the set of all neighbors of v. For $i \ge 2$, the *open* i*-neighborhood* of v, denoted $N_G^i(v)$, is the set of all vertices at distance i from v in G. The *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we simply write V, E, $d(v)$, $N(v)$, $N_2(v)$, $N_3(v)$, $N[v]$, $N(S)$ and $N[S]$ rather than $V(G)$, $E(G)$, $d_G(v)$, $N_G(v)$, $N_G^2(v)$, $N_G^3(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$ respectively.

The *degree* of a vertex v, denoted $d_G(v)$, is the number of neighbors of v, and so $d_G(v) = |N_G(v)|$. A graph G is *subcubic* if its maximum degree is at most 3, and *cubic* if every vertex has degree 3. A vertex of degree 0 is called an *isolated vertex*. For a non-negative integer i, let $n_i(G)$ denote the number of vertices of degree i in a graph G. The *girth* of G, denoted $g(G)$, is the length of a shortest cycle in G. For a set S of vertices in G, the subgraph of G induced by S is denoted by $G[S]$. Further if $S \ne V$, then we denote the graph obtained from G by deleting all vertices in S (as well as all incident edges) by $G - S = G[V(G) \setminus S]$. The *boundary of* S, denoted $\partial(S)$, is the set of all vertices in $V \setminus S$ that have a neighbor in S. For notation and graph theory terminology not defined herein, we refer the reader to [8]. We use the standard notation $[k] = \{1, \dots, k\}$.

* Corresponding author.
E-mail addresses: gh.abrishamimoghadam@mail.um.ac.ir (G. Abrishami), mahenning@uj.ac.za (M.A. Henning).

http://dx.doi.org/10.1016/j.disc.2017.08.025
0012-365X/

ARTICLE IN PRESS

(a) $K_{3,3}$. (b) $C_5 \square K_2$.

Fig. 1. The graphs $K_{3,3}$ and $C_5 \square K_2$.

2. Main result

We shall prove the following result, a proof of which is presented in Section 4.

Theorem 1. *If G is a subcubic graph of girth at least 6, then*

$$11i(G) \le 11n_0(G) + 7n_1(G) + 5n_2(G) + 4n_3(G).$$

As an immediate consequence of Theorem 1, we have our main result.

Theorem 2. *If G is a cubic graph of order n and of girth at least 6, then $i(G) \le \frac{4}{11}n$.*

3. Motivation and known results

It remains an open and challenging problem to determine a sharp upper bound on the independent domination number of a connected, cubic graph, of sufficiently large order, in terms of its order. The best general upper bound to date is due to Lam, Shiu, and Sun [10] who proved that if G is a connected cubic graph G of order n other than $K_{3,3}$, then $i(G) \le \frac{2}{5}n$, where the graph $K_{3,3}$ is given in Fig. 1(a). We remark that this upper bound is achieved by the 5-prism $C_5 \square K_2$ which is depicted in Fig. 1(b).

Goddard and Henning [3] believe that the $\frac{2}{5}n$ bound on $i(G)$ can be improved considerably if we exclude the graphs $K_{3,3}$ and the 5-prism, and posed the following conjecture.

Conjecture 1 ([3]). *If $G \notin \{K_{3,3}, C_5 \square K_2\}$ is a connected, cubic graph of order n, then $i(G) \le \frac{3}{8}n$.*

Dorbec et al. [2] showed that if there is no subgraph isomorphic to $K_{2,3}$, then Conjecture 1 is true.

Theorem 3 ([14]). *If $G \ne C_5 \square K_2$ is a connected, cubic graph of order n that does not have a subgraph isomorphic to $K_{2,3}$, then $i(G) \le \frac{3}{8}n$.*

However, Conjecture 1 remains unresolved in general. Two infinite families of connected, cubic graphs, with independent domination number three-eighths their orders, are constructed in [3], showing that the bound in Conjecture 1, if true, is sharp. We remark that every graph that belongs to one of these two families constructed in [3] contains cycles of length 4 or 5. If we forbid small cycles, then Jacques Verstraete [14] posed the following conjecture.

Conjecture 2 ([14]). *If G is a cubic graph of order n with girth at least 6, then $i(G) \le \frac{1}{3}n$.*

Goddard and Henning [3] posed the following conjecture on the independent domination number of a bipartite, cubic graph.

Conjecture 3 ([3]). *If $G \ne K_{3,3}$ is a connected, bipartite, cubic graph of order n, then $i(G) \le \frac{4}{11}n$.*

Henning, Löwenstein, and Rautenbach [7] proved that Conjecture 3 holds if the graph is quadrilateral-free; that is, they proved that Conjecture 3 is true if the girth is at least 6.

Theorem 4 ([7]). *If G is a cubic bipartite graph of order n and of girth at least 6, then $i(G) \le \frac{4}{11}n$.*

In order to prove Theorem 4, the authors in [7] prove the following stronger result on the independent domination number of a subcubic graph.

Theorem 5 ([7]). *If G is a subcubic bipartite graph of girth at least 6, then*

$$11i(G) \le 11n_0(G) + 7n_1(G) + 5n_2(G) + 4n_3(G).$$