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Neighborhood degree lists of graphs

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ABSTRACT

The neighborhood degree list (NDL) is a graph invariant that refines information given by the degree sequence and joint degree matrix of a graph and is useful in distinguishing graphs having the same degree sequence. We show that the space of realizations of an NDL is connected via a switching operation. We then determine the NDLs that have a unique realization by a labeled graph; the characterization ties these NDLs and their realizations to the threshold graphs and difference graphs.

1. Introduction

Though the degree sequence of a graph is one of the simplest possible invariants of a graph, it has attracted considerable interest and yielded beautiful results. Several different tests are known for determining if a list of integers is a degree sequence, and many authors have written about the properties that the graphs having a given degree sequence (the *realizations* of the sequence) can or must have.

In particular, several authors have asked or answered questions concerning the uniqueness of realizations. In the case of strict uniqueness, where there is only one possible realization of a degree sequence once degrees are prescribed for labeled vertices, the degree sequences involved are the threshold sequences; their realizations are called threshold graphs (see the monograph [16] for a survey). A more relaxed question of uniqueness requires that there can only be one realization of the degree sequence up to isomorphism (the degree sequence $(1, 1, 1, 1)$, for instance, has three distinct realizations but only one up to isomorphism). Degree sequences with realizations from a unique isomorphism class are called unigraphic, and their realizations are unigraphs. For a discussion of unigraphs and a good bibliography, see [21]. (When questions of uniqueness are addressed in later sections of this paper, our understanding of uniqueness will be in the former sense, where isomorphism classes are ignored and graphs with distinct edge sets are considered to be distinct.)

The degree sequence is not the only descriptive parameter based on the degrees of vertices. In [18], Patrinos and Hakimi considered *integer-pair sequences*, collections of unordered pairs of integers produced by recording the degrees of the two endpoints of each edge in a graph (or pseudograph, multigraph, etc.). They determined which sequences of integer pairs can be realized by a graph, pseudograph, or multigraph, and later Das [6] characterized the integer-pair sequences that correspond to a single graph (up to isomorphism). More recently, authors have studied a reformulation of integer-pair sequences known as the *joint degree matrix*, where multiplicities of integer pairs are recorded in matrix form [5,20].

The integer-pair sequence and joint degree matrix yield more information about a graph than a degree sequence does, and in this paper we introduce another degree-related parameter, the *neighborhood degree list* (NDL), that yields still more.

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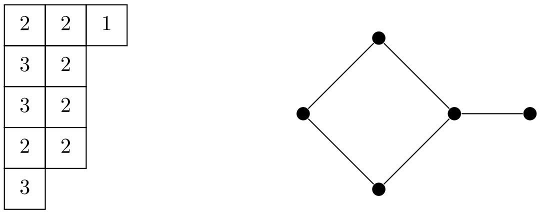

Fig. 1. A graphical depiction of $((2, 2, 1), (3, 2), (3, 2), (2, 2), (3))$.

The neighborhood degree list of a graph G is a list

$$\tau(G) = ((\tau_1^1, \dots, \tau_{d_1}^1), \dots, (\tau_1^n, \dots, \tau_{d_n}^n))$$

whose elements are lists of the degrees in G of the neighbors of a given vertex. For example, if G is the graph obtained by attaching a pendant vertex to a chordless 4-cycle, then $\tau(G) = ((2, 2, 1), (3, 2), (3, 2), (2, 2), (3))$. Notice how the degree sequence $(3, 2, 2, 2, 1)$ of G is apparent from the lengths of the elements of $\tau(G)$ (we call these elements the *component lists*). The order of the main list, together with the order of integers within component lists, is usually of little consequence, though for convenience we will order the integers within a component list from largest to smallest and will list the component lists in descending order of length.

We can represent an NDL graphically by placing the integers it contains into the Young diagram of its degree sequence in much the same way Young tableaux are represented, with the terms of one component list per row. For example, we depict the NDL $((2, 2, 1), (3, 2), (3, 2), (2, 2), (3))$ from above by the diagram in Fig. 1. Note that our orderings of numbers within rows and columns of the diagram does not follow typical monotonicity rules for Young tableaux.

The information contained in neighborhood degree lists was studied by Bassler et al. in [2], in the form of the *degree-spectra matrix*, an incidence matrix with rows indexed by values in the degree sequence and columns indexed by vertices of the graph considered; clearly one can construct the NDL of a graph given its degree-spectra matrix, and vice versa. Degree-spectra matrices were used as an aid in ensemble modeling of graphs with a prescribed joint-degree matrix, and [2] characterizes matrices which are degree-spectra matrices of graphs.

Besides providing more detailed information than the degree sequence and the integer-pair sequence or joint degree matrix, our motivation for studying neighborhood degree lists comes from a few contexts. In situations where it becomes necessary to distinguish between nonisomorphic graphs having the same degree sequence or joint degree matrix, it may be possible to do so by consulting their NDLs. The first author used NDLs in this way in a proof in [1] (see Theorem 2.3 therein); the paper [2] uses degree-spectra matrices for this same purpose.

We find another application in graph reconstruction, which we now describe. The well known Graph Reconstruction Conjecture, attributed to Kelly [13,14] and Ulam [22], states that every n-vertex graph (where $n \geq 3$) is uniquely determined up to isomorphism by the multiset of its induced subgraphs of order $n-1$. These subgraphs are called the *cards* of the graph, and the collection of cards is called the *deck*. Given the deck of an unknown graph G, a standard counting argument yields the degree sequence of G: first, sum the numbers of edges in each of the cards; denote the result by s. The graph G then contains exactly $s/(n-2)$ edges, since each edge appears in all but two of the cards. Given any card of G, we can then subtract the number of edges in the card from the number $s/(n-2)$ to determine the degree of the missing vertex. Doing this for each card in turn yields the degree sequence of G.

However, the degree sequence is not all we can determine in this way. By comparing the degree sequence of the card and the degree of the missing vertex to the degree sequence of the graph G, it is possible to determine the degrees in G of the vertices to which the missing vertex is adjacent. Thus our counting argument yields not only the degree sequence but also the neighborhood degree list of G. (It also shows that the Reconstruction Conjecture is true for graphs that are uniquely determined, up to isomorphism, by their NDLs.)

In this paper we develop foundational results on the NDL of a graph. In Section 2 we characterize the NDLs of simple graphs and describe how to construct the realizations of one. After obtaining the results presented here, the authors learned of the characterization of graphic degree-spectra matrices in [2], which deals with an equivalent problem. However, our solution is different and should provide an alternative approach to this and related questions.

Further in the paper, in Section 3 we present an edge-switching operation for transforming one realization of an NDL into any other realization of the same NDL. Finally, in Section 4, we determine the NDL analogues of threshold sequences and graphs by determining which NDLs have unique realizations by labeled graphs, and which graphs these are.

Throughout the paper, we use $V(G)$ to denote the vertex set of a graph G. We denote the degree sequence of G by $\deg(G)$, and the degree in G of a vertex v by $\deg_G(v)$ or by $\deg(v)$, if G is understood from the context.

2. Feasible tableaux

In this section we characterize those lists of lists of integers that are neighborhood degree lists of simple graphs. We say that a *tableau* is a list

$$T = ((\tau_1^1, \dots, \tau_{d_1}^1), \dots, (\tau_1^n, \dots, \tau_{d_n}^n)) \tag{1}$$

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