Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Diameter bounds for geometric distance-regular graphs

Sejeong Bang

Department of Mathematics, Yeungnam University, 280 Daehak-Ro, Gyeongsan, Gyeongbuk 38541, Republic of Korea

ARTICLE INFO

Article history:
Received 17 November 2016
Received in revised form 23 August 2017
Accepted 28 August 2017
Available online xxxx

Keywords:
Geometric distance-regular graph
Diameter bound
Smallest eigenvalue
Johnson graph
Induced subgraph $K_{2,1,1}$

ABSTRACT

A non-complete distance-regular graph is called geometric if there exists a set $\mathcal{C}$ of Delsarte cliques such that each edge lies in exactly one clique in $\mathcal{C}$. Let Γ be a geometric distance-regular graph with diameter $D \geq 3$ and smallest eigenvalue θ_D. In this paper we show that if Γ contains an induced subgraph $K_{2,1,1}$, then $D \leq -\theta_D$. Moreover, if $-\theta_D - 1 \leq D \leq -\theta_D$ then $D = -\theta_D$ and Γ is a Johnson graph. We also show that for $(s, b) \notin \{(11, 11), (21, 21)\}$, there are no distance-regular graphs with intersection array $\{4s, 3(s-1), s+1-b; 1, 6, 4b\}$ where s, b are integers satisfying $s \geq 3$ and $2 \leq b \leq s$. As an application of these results, we classify geometric distance-regular graphs with $D \geq 3, \theta_D \geq -4$ and containing an induced subgraph $K_{2,1,1}$.

1. Introduction

For a distance-regular graph Γ with valency k, diameter $D \geq 2$ and smallest eigenvalue θ_D, any clique C in Γ satisfies

$$|C| \leq 1 - \frac{k}{\theta_D} \quad (1)$$

(see [5, Proposition 4.4.6(i)]). This bound was shown by Delsarte [6] for strongly regular graphs and Godsil generalized it for distance-regular graphs. A clique of exactly $1 - k/\theta_D$ vertices is called a *Delsarte clique*. The following notion of a geometric distance-regular graph was introduced by Godsil [9]. A non-complete distance-regular graph Γ is called *geometric* if there exists a set $\mathcal{C}$ of Delsarte cliques such that each edge of Γ lies in exactly one clique in $\mathcal{C}$. In this case we say that Γ is *geometric with respect to* $\mathcal{C}$. Note here that for a geometric distance-regular graph, θ_D is integral. There are many examples of geometric distance-regular graphs, such as Hamming graphs, Johnson graphs, Grassmann graphs, bilinear forms graphs, regular near $2D$-gons and bipartite distance-regular graphs.

Let Γ be a geometric distance-regular graph with valency k, diameter $D \geq 2$ and smallest eigenvalue θ_D. Note that a distance-regular graph has no induced subgraph $K_{2,1,1}$ if and only if it is of order (s, t) with $s = a_1 + 1$ and $t = \frac{b_1}{a_1+1}$ (i.e., locally the disjoint union of $t + 1$ cliques of size s). It follows by [2, Theorem 1.4] that for Γ neither bipartite nor complete multipartite, if Γ contains an induced subgraph $K_{2,1,1}$ then $D \leq -\frac{k}{\theta_D}$. In [10, Proposition 4.3], Koolen and Bang showed that if Γ satisfies $c_2 \geq 2$, then $D < (-\theta_D)^2$. In this paper, we show in Theorem 1.1 that if Γ contains an induced subgraph $K_{2,1,1}$, then $D \leq -\theta_D$. Moreover, we prove that if $D \geq 3$ then a Johnson graph of diameter D is the only geometric distance-regular graph satisfying $D \geq -\theta_D - 1$.

E-mail address: sjbang@ynu.ac.kr.

http://dx.doi.org/10.1016/j.disc.2017.08.036

Theorem 1.1. *Fix an integer $m \geq 2$. Suppose that Γ is a geometric distance-regular graph with diameter $D \geq 2$ and smallest eigenvalue $-m$. If Γ contains an induced subgraph $K_{2,1,1}$, then*

$$D \leq m.$$

Moreover, if $D \geq \max\{3, m-1\}$ then $D = m$ and Γ is a Johnson graph.

To prove Theorem 1.1, we first need to consider geometric distance-regular graphs with diameter three. A geometric distance-regular graph with diameter three and smallest eigenvalue $-m$ has intersection array

$$\{ms, (m-1)(s+1-\psi_1), (m-\tau_2)(s+1-\psi_2); 1, \tau_2\psi_1, m\psi_2\}, \tag{2}$$

where $1 \leq \tau_2 < m$ and $1 \leq \psi_1 \leq \psi_2 \leq s$ are all integers (see Lemma 2.1(i)–(ii)). There are many examples of distance-regular graphs with intersection array (2), such as Hamming graphs $H(3, q)$ $(q \geq 2)$, Johnson graphs $J(n, 3)$ $(n \geq 6)$, Grassmann graphs $J_q(n, 3)$ $(n \geq 6)$, bilinear forms graphs $H_q(n, 3)$ and the Doob graph of diameter 3. The Doob graph of diameter 3 is not geometric and it has the same intersection array with the Hamming graph $H(3, 4)$.

Note that any geometric distance-regular graph with diameter three, smallest eigenvalue -3 and $c_2 = 2$ has intersection array (2) with $(m, \tau_2, \psi_1) = (3, 2, 1)$ (cf. Lemma 2.1(iv)). In [3, Theorem 1.3], Bang and Koolen showed that if a distance-regular graph has intersection array (2) with $(m, \tau_2, \psi_1) = (3, 2, 1)$ and $1 < \psi_2 < s$, then $(s, \psi_2) = (15, 9)$. Using together with results [1, Theorem 4.3] and [7], this enables us to classify geometric distance-regular graphs with diameter $D \geq 3$, smallest eigenvalue -3 and $c_2 \geq 2$ (see [3, Theorem 1.4]).

Note that any geometric distance-regular graph with diameter three, smallest eigenvalue -4 and $c_2 = 6$ has intersection array (2) with $(m, \tau_2, \psi_1) = (4, 3, 2)$ (i.e., it has intersection array (3)). We show in Theorem 1.2 that there are no distance-regular graphs with intersection array (3) with a possible exception when $(s, b) \in \{(11, 11), (21, 21)\}$. In particular, if $(s, b) \in \{(11, 11), (21, 21)\}$ then it is not geometric. Using Theorem 1.2, we show in Theorem 1.1 that there are no geometric distance-regular graphs with diameter $D = -\theta_D - 1 \geq 3$ and containing an induced subgraph $K_{2,1,1}$.

Theorem 1.2. *If Γ is a distance-regular graph with intersection array*

$$\{4s, 3(s-1), s+1-b; 1, 6, 4b\} \quad \text{where } s \geq 3 \text{ and } 2 \leq b \leq s \tag{3}$$

then $(s, b) \in \{(11, 11), (21, 21)\}$. In particular, Γ is antipodal but not geometric.

Neumaier [12] showed that for a given integer $m \geq 2$, except for a finite number of graphs, any geometric strongly regular graph with smallest eigenvalue $-m$ is either a Steiner graph or a Latin square graph. There are no geometric distance-regular graphs with diameter $D \geq 3$, smallest eigenvalue -2 and $c_2 \geq 2$ (see [5, Theorems 3.12.2 and 4.2.16]). Bang [1, Theorem 4, 3] showed that the Johnson graph $J(n, 3)$ $(n \geq 6)$ is the only geometric distance-regular graph with diameter $D \geq 3$ and smallest eigenvalue -3 which contains an induced subgraph $K_{2,1,1}$. This result can be obtained by Theorem 1.1 with $m = 3$. Using Theorem 1.1, we show the following result for $\theta_D \geq -4$.

Corollary 1.3. *Suppose that Γ is a geometric distance-regular graph with diameter $D \geq 3$ and smallest eigenvalue at least -4. If Γ contains an induced subgraph $K_{2,1,1}$, then Γ is the Johnson graph $J(n, D)$ $(n \geq 2D)$, where $D \in \{3, 4\}$.*

This paper is organized as follows. In Section 2, we review some notations and basic concepts. In Section 3, we prove Theorem 1.2, which implies that there are no geometric distance-regular graphs with diameter three, smallest eigenvalue -4 and $c_2 = 6$. To show Theorem 1.2 we consider two cases, $s \leq 35$ and $s > 35$. We prove that if $s \leq 35$ then $(s, b) \in \{(11, 11), (21, 21)\}$ and Γ is not geometric for these two cases (see Lemmas 3.1 and 3.2). On the other hand, if $s > 35$ then we first show that Γ must be geometric by using Proposition 3.4, and Γ has only integral eigenvalues (see Lemma 3.3). Using Lemmas 3.2 and 3.3 we prove that there are no geometric distance-regular graphs with intersection array (3) with $s > 35$. In Section 4, we prove Theorem 1.1 and Corollary 1.3. In Lemma 4.1 we show that for a non-complete geometric distance-regular graph, if $\psi_1 \geq 2$ then the sequence $(\tau_i)_{1 \leq i \leq D}$ is strictly increasing. Using this we show in Theorem 1.1 that if a non-complete geometric distance-regular graph Γ contains an induced subgraph $K_{2,1,1}$ then $D \leq -\theta_D$. In particular, we show in Theorem 1.1 that if $D = -\theta_D \geq 3$ then Γ is a Johnson graph, and there are no geometric distance-regular graphs with $D = -\theta_D - 1 \geq 3$ by using Theorem 1.2. As an application of Theorem 1.1, we classify geometric distance-regular graphs with $D \geq 3$ and $\theta_D \geq -4$ which are not locally a disjoint union of cliques (Corollary 1.3).

2. Preliminaries

All graphs in this paper are finite, undirected and simple. Let Γ be a connected graph. For any two vertices x, y in the vertex set $V(\Gamma)$ of Γ, the distance $d(x, y)$ between x and y is the length of a shortest path between them in Γ, and the diameter D is the maximum distance between any two vertices of Γ. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x) := \{z \in V(\Gamma) \mid d(x, z) = i\}$ $(0 \leq i \leq D)$. In addition, define $\Gamma_{-1}(x) = \emptyset$ and $\Gamma_{D+1}(x) = \emptyset$. Let $v := |V(\Gamma)|$. The adjacency matrix A_Γ of Γ is the $(v \times v)$-matrix with rows and columns are indexed by $V(\Gamma)$, where the (x, y)-entry of A_Γ is 1 if $d(x, y) = 1$ and 0 otherwise.