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Acyclic subgraphs with high chromatic number ☆

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ABSTRACT

For an oriented graph G, let $f(G)$ denote the maximum chromatic number of an acyclic subgraph of G. Let $f(n)$ be the smallest integer such that every oriented graph G with chromatic number larger than $f(n)$ has $f(G) > n$. Let $g(n)$ be the smallest integer such that every tournament G with more than $g(n)$ vertices has $f(G) > n$. It is straightforward that $\Omega(n) \leq g(n) \leq f(n) \leq n^2$. This paper provides the first nontrivial lower and upper bounds for $g(n)$. In particular, it is proved that $\frac{1}{4}n^{8/7} \leq g(n) \leq n^2 - (2 - \frac{1}{\sqrt{2}})n + 2$. It is also shown that $f(2) = 3$, i.e. every orientation of a 4-chromatic graph has a 3-chromatic acyclic subgraph. Finally, it is shown that a random tournament G with n vertices has $f(G) = \Theta(\frac{n}{\log n})$ whp.

1. Introduction

All graphs in this paper are finite and simple. An *orientation* of an undirected graph is obtained by assigning a direction to each edge. An important class of oriented graphs are *tournaments* which are orientations of a complete graph. We denote by T_n the unique acyclic (thereby transitive) tournament with n vertices. An acyclic subgraph of an oriented graph is a subgraph having no directed cycles. In this paper, the chromatic number of an oriented graph is the chromatic number of its underlying undirected graph.

It is a folklore argument that every oriented graph has acyclic subgraphs containing at least half of the edges. Indeed, every linear ordering of the vertices partitions the edge set to two acyclic subgraphs, one consisting of the edges pointing from lower vertices to higher vertices and the other consisting of the edges pointing from higher vertices to lower vertices. At least one of these two subgraphs contains at least half of the edges. Thus, we are guaranteed to find dense (with respect to the density of the original oriented graph) acyclic subgraphs. However, apart from density, we do not have much control

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on other complexity parameters of these acyclic subgraphs. Perhaps the most natural is the following question raised by Addario-Berry et al. [1] and also in the survey of Havet [9]. Suppose we know that the chromatic number of our oriented graph is large, can we guarantee that an acyclic subgraph of it also has high chromatic number?

Definition 1.1. For an oriented graph G, let $f(G)$ denote the maximum chromatic number of an acyclic subgraph of G. Let $f(n)$ be the smallest integer such that every oriented graph G with chromatic number larger than $f(n)$ has $f(G) > n$.

It was observed in [1] that $f(n) \leq n^2$ by the following standard "product-coloring" argument. Suppose G is a graph, and take any partition of its edge set into k parts, inducing subgraphs $G_1, \ldots, G_k$. Then clearly, $\prod_{i=1}^{k} \chi(G_i) \geq \chi(G)$ as we may properly color each vertex of G by the k-dimensional vector whose i'th entry is the color that vertex received in a coloring of G_i with $\chi(G_i)$ colors. In particular, for the case where G is an oriented graph with $\chi(G) \geq n^2+1$ and G_1 and G_2 are acyclic subgraphs forming a partition of the edge set according to some linear ordering of the vertices as described in the previous paragraph, then $\chi(G_1) \cdot \chi(G_2) \geq n^2 + 1$ so at least one of them has chromatic number at least $n + 1$. This shows that $f(n) \leq n^2$. To date, there is no known improvement over this simple upper bound.

What if we restrict the question to tournaments? A tournament with $g(n)$ vertices has chromatic number $g(n)$ as it is an orientation of $K_{g(n)}$. Thus, we have the following definition.

Definition 1.2. Let $g(n)$ be the smallest integer such that every tournament G with more than $g(n)$ vertices has $f(G) > n$.

Clearly, we have $g(n) \leq f(n)$ and hence the aforementioned simple upper bound $g(n) \leq n^2$ holds here as well. Our first result is a modest, yet nontrivial improvement to the upper bound.

Theorem 1.3. $g(n) \leq n^2 - (2 - \frac{1}{\sqrt{2}})n + 2$.

We suspect that this upper bound can be improved and raise the following conjecture.

Conjecture 1.4. $g(n) = o(n^2)$.

As for $f(n)$, while we cannot improve upon the upper bound $f(n) \leq n^2$ in general, we do settle the first nontrivial case.

Theorem 1.5. *Suppose G is an orientation of a 4-chromatic graph. Then G has an acyclic subgraph with chromatic number at least 3. In particular, $f(2) = 3$.*

Notice that $g(2) = 3$ is trivial since every tournament with more than 3 vertices has an acyclic triangle and since each acyclic subgraph of the directed triangle has chromatic number at most 2. Whether $f(n) = g(n)$ for larger n remains open.

Let $h(n)$ be the least integer such that every oriented graph with chromatic number $h(n)$ contains every oriented tree with n vertices. A longstanding conjecture of Burr [4] asserts that $h(n) = 2n - 2$. Presently the best known upper bound is $h(n) \leq n^2/2 - n/2 + 1$ given in [1]. One motivation for studying $f(n)$, given in [1], stems from the fact proved there that every acyclic oriented graph with chromatic number n contains every oriented tree on n vertices. Hence, any upper bound for $f(n)$ can be used as an upper bound for $h(n)$ and any lower bound for $f(n)$ which is significantly larger than linear shows that obtaining improvements for $h(n)$ are limited with this approach. Our next result provides such a lower bound for $g(n)$ and hence a lower bound for $f(n)$.

Theorem 1.6. *There are tournaments G with more than $n^{8/7}/4$ vertices such that $f(G) \leq n$. Consequently, $g(n) \geq n^{8/7}/4$.*

In spite of the fact proved in Theorem 1.6 that there are tournaments that have the property that every acyclic subgraph has chromatic number polynomially smaller than the number of vertices of the tournament, it turns out that almost all tournaments do have acyclic subgraphs with only a logarithmic fraction loss in the chromatic number. Recall that the random tournament $\mathcal{G}(n)$ is the probability