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Bipartite and Eulerian minors

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ABSTRACT

This paper defines the related notions of bipartite and Eulerian minors for binary matroids. Using these definitions, it characterizes graphic matroids within the classes of bipartite binary matroids and Eulerian binary matroids by the exclusion of certain bipartite minors and Eulerian minors, respectively. This result on Eulerian minors in binary matroids extends a result of Chudnovsky et al. who characterized planar graphs within the class of bipartite graphs by the exclusion of $K_{3,3}$ as a bipartite minor.

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1. Introduction

Bipartite graphs are not closed under the usual graph minor operations. Specifically, contracting an edge of a bipartite graph can produce a non-bipartite graph. Thus, any interesting subclass of bipartite graphs, such as the class of planar bipartite graphs, cannot be characterized using the usual definition of a graph minor. To circumvent this, Chudnovsky et al. [3] introduced a new minor operation, which they called "bipartite contraction". By replacing the usual contraction operation with bipartite contraction, they showed that the subclass of bipartite planar graphs can be characterized by the exclusion of $K_{3,3}$ as a so-called "bipartite minor".

The main purpose of this paper is to extend the definition of bipartite minors, as well as the related notion of Eulerian minors, to binary matroids, and then, using these definitions, prove the following two theorems.

Theorem 1. *(a) An Eulerian binary matroid is graphic if and only if it does not contain either $M^*(K_{3,3})$ or F_7 as an Eulerian minor, and (b) a bipartite binary matroid is cographic if and only if it does not contain either $M(K_{3,3})$ or F_7^* as a bipartite minor.*

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Theorem 2. *(a) A bipartite binary matroid is graphic if and only if it does not contain any of $M^*(K_5)$, $M^*(\widehat{K}_{3,3})$, or F_7^* as a bipartite minor, and (b) an Eulerian binary matroid cographic if and only if it does not contain any of $M(K_5)$, $M(\widehat{K}_{3,3})$, or F_7 as an Eulerian minor.*

As will be seen, in each theorem, the second statement is just the dual of the first. In Theorem 2, $\widehat{K}_{3,3}$ is the graph obtained from $K_{3,3}$ by "doubling" each edge in some perfect matching. Theorem 1 can be seen as a generalization of the result of Chudnovsky et al. on bipartite graphs, and similarly Theorem 2 is a generalization of an analogous result, Theorem 4 below, on Eulerian graphs. Theorems 1 and 2 were independently proved by Chun and Oxley [4].

The remainder of this paper is organized as follows. The next section introduces "pinches" and "splits", which are the matroid operations used to define bipartite and Eulerian minors, respectively. Section 3 contains results on bipartite and Eulerian graphs, and Section 4 contains results on bipartite and Eulerian matroids. The proof of Theorems 1 and 2 are then contained in Section 5.

2. Pinches and splits

Undefined graph and matroid terminology is standard; see, for example, Oxley [7].

A connected graph is *nonseparable* if every pair of edges are contained in a cycle. A cycle C of a nonseparable graph G is *nonseparating* if the graph $G/E(C)$ is nonseparable. A cycle C of a nonseparable graph G is *almost nonseparating* if it is a nonseparating cycle in the graph obtained from G by first deleting all of the edges of $E(G)-E(C)$ that are parallel to an edge of C. A *star* of a node v in a graph G is the set of edges of G incident to v. A *cocycle* of a connected graph is a minimal set of edges, the deletion of which produces a disconnected graph. A cocycle D of a nonseparable graph G is *nonseparating* if the graph $G\backslash D$ is nonseparable. Observe that a nonseparating cocycle of G is necessarily the star of some node of G. A cocycle D of a nonseparable graph G is *almost nonseparating* if it is a nonseparating cocycle in the graph obtained from G by first contracting all of the edges of $E(G) - E(D)$ that are in series with an edge of D.

Let G be a nonseparable graph, and let e and f be a pair of adjacent, but not parallel, edges of G. Let x and y denote the non-common end nodes of e and f, respectively. An $\{e, f\}$-*pinch* of G is the operation on G defined by identifying nodes x and y; the resulting graph is denoted $G \wedge \{e, f\}$. More generally, a *pinch* of a graph G is an $\{e, f\}$-pinch of G for some non-parallel pair $\{e, f\}$ of adjacent edges of G. An $\{e, f\}$-pinch of a nonseparable graph G is called *admissible* if there exists an almost-nonseparating cycle and an almost-nonseparating cocycle of G, both of which contain e and f. A pinch in a bipartite graph is called a *bipartite contraction* in Chudnovsky et al. [3].

Again, let G be a nonseparable graph, and now let e and f be edges incident to node v having degree at least three. An $\{e, f\}$-*split* of G is the operation defined by first adding a new node, say z, to G, and then re-defining the node–edge incidence relationships of e and f so that they are now incident to z instead of v (while all other node–edge incidence relationships remained unchanged); the resulting graph is denoted $G \vee \{e, f\}$. (To be precise, in the case that e and f are parallel, $G \vee \{e, f\}$ is not well defined unless one also specifies the node v.) More generally, a *split* of a graph G is an $\{e, f\}$-split of G for some pair $\{e, f\}$ of edges incident to a node of degree of at least three of G. An $\{e, f\}$-split of a nonseparable graph G is called *admissible* if there exists an almost-nonseparating cycle and an almost-nonseparating cocycle of G, both of which contain e and f.

Now, consider the case when G is a bipartite graph, and let e and f be a pair of non-parallel, adjacent edges. Observe that the graph obtained by an $\{e, f\}$-pinch in G is also bipartite. A graph H is a *bipartite minor* of G if there exists a sequence of graphs taking G to H such that each graph in the sequence is obtained from its predecessor by either an edge deletion, an admissible pinch, or the deletion of an isolated node. Thus, a bipartite minor of a bipartite graph is bipartite.

Now, assume that G is an Eulerian graph, and let e and f be edges incident to a node of degree at least four. Then, observe that the graph obtained by an $\{e, f\}$-split is also Eulerian. A graph H is an *Eulerian minor* of G if there exists a sequence of graphs taking G to H such that each graph in the sequence is obtained from its predecessor by either an edge contraction, an admissible split, or the deletion of an isolated node. Thus, an Eulerian minor of an Eulerian graph is Eulerian.

The discussion now turns to matroids, specifically bipartite and Eulerian matroids, which were introduced by Welsh [12]. Welsh defined a matroid to be *bipartite* if every circuit has even cardinality,