European Journal of Combinatorics 69 (2018) 255–259

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

A note on minimal dispersion of point sets in the unit cube ☆

Jakub Sosnovec

Department of Computer Science, University of Warwick, United Kingdom

ARTICLE INFO

Article history:
Received 29 July 2017
Accepted 13 November 2017
Available online 6 December 2017

ABSTRACT

We study the dispersion of a point set, a notion closely related to the discrepancy. Given a real $r \in (0, 1)$ and an integer $d \geq 2$, let $N(r, d)$ denote the minimum number of points inside the d-dimensional unit cube $[0, 1]^d$ such that they intersect every axis-aligned box inside $[0, 1]^d$ of volume greater than r. We prove an upper bound on $N(r, d)$, matching a lower bound of Aistleitner et al. up to a multiplicative constant depending only on r. This fully determines the rate of growth of $N(r, d)$ if $r \in (0, 1)$ is fixed.

1. Introduction

The geometric discrepancy theory is the study of distributions of finite point sets and their irregularities [5]. In this note, we study a notion closely related to discrepancy, the dispersion of a point set.

The problem of finding the area of the largest empty axis-parallel rectangle amidst a set of points in the unit square is a classical problem in computational geometry. The algorithmic version has been introduced by Naamad et al. [6] and several other algorithms have been proposed over the years, such as [2]. The problem naturally generalizes to multi-dimensional variant, where the task is to determine the volume of the largest empty box amidst a set of points in the d-dimensional unit cube.

An active line of research concerns general bounds on the volume of largest empty box for *any* set of points, in terms of the dimension and the number of points. An upper bound thus amounts to exhibiting an example of a point set such that the volume of any empty box is small, while the lower bound asks for the minimal value such that every set of points of given cardinality allows an empty box of that volume. The first results in this direction were given by Rote and Tichy [9]. Dumitrescu and Jiang [4] first showed a non-trivial lower bound, which was later improved by Aistleitner et al. [1]. An

☆ This work was supported by the Leverhulme Trust 2014 Philip Leverhulme Prize of Daniel Král'.
E-mail address: j.sosnovec@warwick.ac.uk.

https://doi.org/10.1016/j.ejc.2017.11.006
0195-6698/

upper bound by Larcher is also given in [1]. Rudolf [10] has found an upper bound with much better dependence on the dimension. The problem has recently received attention due to similar questions appearing in approximation theory [8], discrepancy theory [3,7] and approximation of L_p-norms and Marcinkiewicz-type discretization [12,11,13].

The following reformulation is of interest in the applications in approximation theory: If we fix $r \in (0, 1)$ to be the "allowed volume", how many points in $\mathbb{R}^d$ are needed to force that any empty box has volume at most r, in terms of d? In other words, we ask for the minimum number of points needed to intersect every box of volume greater than r. In this note, we establish the optimal asymptotic growth of this quantity for r fixed.

1.1. Notation

For a positive integer k, let $[k]$ denote the set $\{1, 2, \dots, k\}$. If $x \in \mathbb{R}^d$ is a vector, $(x)_i$ denotes the ith coordinate of x. Let $\mathbf{1}$ denote the vector $(1, \dots, 1) \in \mathbb{R}^d$ (where d will always be clear from context).

For $d \geq 2$, we use $[0, 1]^d$ to denote the d-dimensional unit cube. A box $B = I_1 \times \cdots \times I_d \subseteq [0, 1]^d$ is *open* (*closed*) if all of $I_1, \dots, I_d$ are open (closed) intervals. We define $\mathscr{B}_d$ as the family of all open boxes inside $[0, 1]^d$.

For a set T of n points in $[0, 1]^d$, the volume of the largest open axis-parallel box avoiding all points from T is called the *dispersion* of T and is defined as

$$\mathrm{disp}(T) = \sup_{B \in \mathscr{B}_d,\, B \cap T = \emptyset} \mathrm{vol}(B), \tag{1}$$

where $\mathrm{vol}(I_1 \times \cdots \times I_d) = |I_1| \cdots |I_d|$. Note that the supremum in (1) is attained, since there are only finitely many inclusion-maximal boxes $B \in \mathscr{B}_d$ avoiding T.

We further define the minimal dispersion for any point set as

$$\mathrm{disp}^*(n, d) = \inf_{T \subset [0,1]^d,\, |T| = n} \mathrm{disp}(T). \tag{2}$$

Again, observe that the infimum in (2) is actually attained, since any sequence of n-element point sets inside $[0, 1]^d$ has a convergent subsequence.

The quantity we mainly consider in this paper is the inverse function of the minimal dispersion,

$$N(r, d) = \min\{n \in \mathbb{N} : \mathrm{disp}^*(n, d) \leq r\},$$

where $r \in (0, 1)$. Determining $N(r, d)$ thus corresponds to the question of how many points are needed to intersect every box of volume greater than r.

We remark that the functions $\mathrm{disp}^*(n, d)$ and $N(r, d)$ are of course tightly connected and any bounds on them translate between each other.

1.2. Previous work

The trivial lower bound on $\mathrm{disp}^*(n, d)$ is $1/(n + 1)$, since we can split the cube into $n + 1$ parts and use the pigeonhole principle. This was improved in [1] to

$$\mathrm{disp}^*(n, d) \geq \frac{\log_2 d}{4(n + \log_2 d)}. \tag{3}$$

The inequality (3) can be reformulated to give a lower bound on $N(r, d)$ for $r \in (0, 1/4)$,

$$N(r, d) \geq \frac{1 - 4r}{4r} \log_2 d. \tag{4}$$

In order to show (3), the same authors prove an auxiliary lemma, which is equivalent to that

$$N(1/4, d) \geq \log_2(d + 1).$$

Thus for $r \in (0, 1/4]$ fixed, we have $N(r, d) = \Omega(\log d)$.