Journal of Combinatorial Theory, Series A 161 (2019) 51–80

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

On ranks and cranks of partitions modulo 4 and 8

Eric T. Mortenson

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

ARTICLE INFO

Article history:
Received 13 December 2017
Available online xxxx

Keywords:
Partitions
Rank
Crank
Mock theta functions

ABSTRACT

Denote by $p(n)$ the number of partitions of n and by $N(a, M; n)$ the number of partitions of n with rank congruent to a modulo M. By considering the deviation

$$D(a, M) := \sum_{n=0}^{\infty}\left(N(a, M; n) - \frac{p(n)}{M}\right)q^n,$$

we give new proofs of recent results of Andrews, Berndt, Chan, Kim and Malik on mock theta functions and ranks of partitions. By considering deviations of cranks, we give new proofs of Lewis and Santa-Gadea's rank-crank identities. We revisit ranks and cranks modulus $M = 5$ and 7, with our results on cranks appearing to be new. We also demonstrate how deviations of ranks and cranks resolve Lewis' long-standing conjectures on identities and inequalities for rank-crank differences of modulus $M = 8$.

E-mail address: etmortenson@gmail.com.

https://doi.org/10.1016/j.jcta.2018.07.009
0097-3165/

0. Notation

Let q be a complex number with $0<|q|<1$ and define $\mathbb{C}^* := \mathbb{C}-\{0\}$. For $n\geq 0$,

$$(x)_\infty = (x;q)_\infty := \prod_{i=0}^{\infty}(1-q^ix), \quad (x)_n = (x;q)_n := \frac{(x)_\infty}{(q^nx)_\infty},$$

$$\text{and} \quad j(x;q) := (x)_\infty (q/x)_\infty (q)_\infty = \sum_{k=-\infty}^{\infty}(-1)^k q^{\binom{k}{2}} x^k,$$

where in the last line the equivalence of product and sum follows from Jacobi's triple product identity. Let a and m be integers with m positive. Define

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m), \text{ and } J_m := J_{m,3m} = \prod_{i=1}^{\infty}(1-q^{mi}).$$

1. Introduction

We recall a universal mock theta function

$$g(x;q) := x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x)_{n+1}(q/x)_n}\right). \tag{1.1}$$

One of the earliest celebrated results in the history of mock theta functions was Hickerson's proof of the mock theta conjectures, that express fifth order mock theta functions $f_0(q)$ and $f_1(q)$ in terms of the universal mock theta function $g(x;q)$.

Theorem 1.1 *([8]). We have*

$$f_0(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n} = -2q^2 g(q^2;q^{10}) + \frac{J_{5,10}J_{2,5}}{J_1},$$

$$f_1(q) := \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n} = -2q^3 g(q^4;q^{10}) + \frac{J_{5,10}J_{1,5}}{J_1}.$$

Mock theta functions and the study of partitions are inextricably linked. A partition of a positive integer n is a weakly-decreasing sequence of positive integers whose sum is n. For example the partitions of the number 4 are (4), $(3,1)$, $(2,2)$, $(2,1,1)$, $(1,1,1,1)$. We denote the number of partitions of n by $p(n)$. Among the most famous results in the theory of partitions are Ramanujan's congruences:

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$