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An upper bound on the size of diamond-free families of sets

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ABSTRACT

Let $\mathrm{La}(n, P)$ be the maximum size of a family of subsets of $[n] = \{1, 2, \ldots, n\}$ not containing P as a (weak) subposet. The diamond poset, denoted $\mathcal{Q}_2$, is defined on four elements x, y, z, w with the relations $x < y, z$ and $y, z < w$. $\mathrm{La}(n, P)$ has been studied for many posets; one of the major open problems is determining $\mathrm{La}(n, \mathcal{Q}_2)$. It is conjectured that $\mathrm{La}(n, \mathcal{Q}_2) = (2 + o(1))\binom{n}{\lfloor n/2 \rfloor}$, and infinitely many significantly different, asymptotically tight constructions are known.

Studying the average number of sets from a family of subsets of $[n]$ on a maximal chain in the Boolean lattice $2^{[n]}$ has been a fruitful method. We use a partitioning of the maximal chains and introduce an induction method to show that $\mathrm{La}(n, \mathcal{Q}_2) \leq (2.20711 + o(1))\binom{n}{\lfloor n/2 \rfloor}$, improving on the earlier bound of $(2.25 + o(1))\binom{n}{\lfloor n/2 \rfloor}$ by Kramer, Martin and Young.

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1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$. The Boolean lattice $2^{[n]}$ is defined as the family of all subsets of $[n] = \{1, 2, \ldots, n\}$, and the *ith level* of $2^{[n]}$ refers to the collection of all sets of size i. In 1928, Sperner proved the following well-known theorem.

Theorem 1.1 *(Sperner [24]). If $\mathcal{F}$ is a family of subsets of $[n]$ such that no set contains another ($A, B \in \mathcal{F}$ implies $A \not\subset B$), then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. Moreover, equality occurs if and only if $\mathcal{F}$ is a level of maximum size in $2^{[n]}$.*

Definition 1.2. Let P be a finite poset, and $\mathcal{F}$ be a family of subsets of $[n]$. We say that P is contained in $\mathcal{F}$ as a (weak) subposet if there is an injection $\varphi : P \to \mathcal{F}$ satisfying $x_1 <_p x_2 \Rightarrow \varphi(x_1) \subset \varphi(x_2)$ for every $x_1, x_2 \in P$. $\mathcal{F}$ is called P-free if P is not contained in $\mathcal{F}$ as a weak subposet. We define the corresponding extremal function as $\mathrm{La}(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } P\text{-free}\}$.

A k-chain, denoted by P_k, is defined to be the poset on the set $\{x_1, x_2, \ldots, x_k\}$ with the relations $x_1 \leq x_2 \leq \cdots \leq x_k$. Using the above notation, Sperner's theorem can be stated as $\mathrm{La}(n, P_2) = \binom{n}{\lfloor n/2 \rfloor}$. Let $\Sigma(n, k)$ denote the sum of the k largest binomial coefficients of order n. An important generalization of Sperner's theorem due to Erdős [10] states that $\mathrm{La}(n, P_{k+1}) = \Sigma(n, k)$. Moreover, equality occurs if and only if $\mathcal{F}$ is the union of k of the largest levels in $2^{[n]}$.

Definition 1.3 *(Posets $\mathcal{Q}_2, V$ and Λ).* The diamond poset, denoted $\mathcal{Q}_2$ (or $\mathcal{D}_2$ or $\mathcal{B}_2$), is a poset on four elements $\{x, y, z, w\}$, with the relations $x < y, z$ and $y, z < w$. That is, $\mathcal{Q}_2$ is a subposet of a family of sets $\mathcal{A}$ if there are different sets $A, B, C, D \in \mathcal{A}$ with $A \subset B, C$ and $B, C \subset D$. (Note that B and C are not necessarily unrelated.) The V poset is a poset on $\{x, y, z\}$ with the relations $x \leq y, z$; the Λ poset is defined on $\{x, y, z\}$ with the relations $x, y \leq z$. That is, the Λ is a subposet of a family of sets $\mathcal{A}$ if there are different sets $B, C, D \in \mathcal{A}$ with $B, C \subset D$.

The general study of forbidden poset problems was initiated in the paper of Katona and Tarján [16] in 1983. They determined the size of the largest family of sets containing neither a V nor a Λ. They also gave an estimate on the maximum size of V-free families: $\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor n/2 \rfloor} \leq \mathrm{La}(n, V) \leq \left(1 + \frac{2}{n}\right)\binom{n}{\lfloor n/2 \rfloor}$. This result was later generalized by De Bonis and Katona [8] who obtained bounds for the r-fork poset, V_r defined by the relations $x \leq y_1, y_2, \ldots, y_r$. Other posets for which $\mathrm{La}(n, P)$ has been studied include complete two level posets, batons [25], crowns O_{2k} (cycle of length $2k$ on two levels, asymptotically solved except for $k \in \{3, 5\}$ [13,18]), butterfly [9], skew-butterfly [22], the N poset [11], harp posets $\mathcal{H}(l_1, l_2, \ldots, l_k)$, defined by k chains of length l_i between two fixed elements [14], and recently the complete 3 level poset $K_{r,s,t}$ [23] among others. (See [12] for a nice survey by Griggs and Li.)