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Explicit bounds for graph minors ☆

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ABSTRACT

Let Σ be a surface with boundary $\mathsf{bd}(\Sigma)$, $\mathcal{L}$ be a collection of k disjoint $\mathsf{bd}(\Sigma)$-paths in Σ, and P be a non-separating $\mathsf{bd}(\Sigma)$-path in Σ. We prove that there is a homeomorphism $\phi : \Sigma \to \Sigma$ that fixes each point of $\mathsf{bd}(\Sigma)$ and such that $\phi(\mathcal{L})$ meets P at most $2k$ times.

With this theorem, we derive explicit constants in the graph minor algorithms of Robertson and Seymour (1995) [10]. We reprove a result concerning redundant vertices for graphs on surfaces, but with explicit bounds. That is, we prove that there exists a *computable* integer $t := t(\Sigma, k)$ such that if v is a 't-protected' vertex in a surface Σ, then v is redundant with respect to any k-linkage.

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1. Introduction

In [12], Robertson and Seymour prove the remarkable theorem that every minor-closed property of graphs is characterized by a finite set of excluded minors.

Theorem 1.1. *For every minor-closed class of graphs* $\mathcal{C}$*, there exists a finite set of graphs* $\mathsf{ex}(\mathcal{C})$*, such that a graph is in* $\mathcal{C}$ *if and only if it does not contain a minor isomorphic to a member of* $\mathsf{ex}(\mathcal{C})$*.*

Robertson and Seymour also prove an important algorithmic counterpart to this theorem in [10,13].

Theorem 1.2. *For any fixed graph* H*, there exists a polynomial-time algorithm to test if an input graph* G *contains a minor isomorphic to* H*.*

Together, these two theorems imply that there *exists* a polynomial-time algorithm to test for membership in any minor-closed class of graphs. Of course, the existence of such an algorithm is highly non-constructive as $\mathsf{ex}(\mathcal{C})$ is explicitly known for only a few minor-closed classes $\mathcal{C}$.

The running time of the algorithm from [10] depends on a function $t(k, \Sigma)$ for irrelevant vertices for k-linkage problems in a surface Σ. Robertson and Seymour clearly state that $t(k, \Sigma)$ is computable, but give no indication how to compute it. In the special case that Σ is the sphere, Adler, Kolliopoulos, Krause, Lokshtanov, Saurabh, and Thilikos [1] do obtain an explicit function (of k).

In addition, Kawarabayashi and Wollan [3] recently gave a simpler algorithm and shorter proof for the powerful graph minor decomposition theorem in [11]. Their approach yields explicit constants for the decomposition algorithm, but again implicitly assumes that $t(k, \Sigma)$ is computable.

In this paper, we show that $t(k, \Sigma)$ is indeed computable, thereby obtaining explicit bounds for graph minors. Before stating our main theorems, we require a few definitions. In this work we use $\Sigma(a, b, c)$ to denote the surface that is the (2-dimensional) sphere with a handles, b crosscaps, and c boundary components, which we call *holes*. We set $g(\Sigma(a, b, c)) := 2a + b$ and $\mathsf{holes}(\Sigma(a, b, c)) = c$.

A *curve* γ in a surface Σ is a continuous function $\gamma : [0, 1] \to \Sigma$. A curve γ

- has *ends* $\gamma(0)$ and $\gamma(1)$;
- is a *path* if it is injective (or constant);
- is a *simple closed curve* if $\gamma(0) = \gamma(1)$ and is injective on $(0, 1]$;
- is *separating* if $\Sigma - \gamma([0, 1])$ is disconnected and *non-separating* otherwise.

Let $X \subseteq \Sigma$.

- The boundary and interior of X will be denoted $\mathsf{bd}(X)$ and $\mathsf{int}(X)$, respectively.

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