ARTICLE IN PRESS YJCTB:3134

Journal of Combinatorial Theory, Series B ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Bipartitions of oriented graphs ☆

Jianfeng Hou [a], Shufei Wu [a,b]

[a] *Center for Discrete Mathematics, Fuzhou University, Fujian, 350116, China*
[b] *School of Mathematics and Information Science, Henan Polytechnic University, Henan, 454000, China*

A R T I C L E I N F O

Article history:
Received 12 July 2016
Available online xxxx

Keywords:
Oriented graph
Partition
Semidegree
Tight component

A B S T R A C T

Let $V(D) = X \cup Y$ be a bipartition of a directed graph D. We use $e(X, Y)$ to denote the number of arcs in D from X to Y. Motivated by a conjecture posed by Lee, Loh and Sudakov (2016) [16], we study bipartitions of oriented graphs. Let D be an oriented graph with m arcs. In this paper, it is proved that if the minimum degree of D is δ, then D admits a bipartition $V(D) = V_1 \cup V_2$ such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \big(\frac{\delta-1}{4\delta} + o(1)\big)m$. Moreover, if the minimum semidegree $d = \min\{\delta^+(D), \delta^-(D)\}$ of D is at least 21, then D admits a bipartition $V(D) = V_1 \cup V_2$ such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \big(\frac{d}{2(2d+1)} + o(1)\big)m$. Both bounds are asymptotically best possible.

1. Introduction

An example of classical graph partitioning problems is the well-known *Max-Cut Problem*: Given a graph G, find the maximum bipartite subgraph of G. A simple average argument shows that every graph with m edges has a bipartite subgraph with at least $m/2$ edges. In 1973, answering a question of Erdős, Edwards [5,6] improved this lower

☆ This work is supported by the National Natural Science Foundation of China (No. 11671087).
E-mail address: shufeiwu@hotmail.com (S. Wu).

https://doi.org/10.1016/j.jctb.2018.03.003
0095-8956/

Please cite this article in press as: J. Hou, S. Wu, Bipartitions of oriented graphs, J. Combin. Theory Ser. B (2018), https://doi.org/10.1016/j.jctb.2018.03.003

bound to $m/2+(\sqrt{2m+1/4}-1/2)/4$, which is essentially best possible as evidenced by complete graphs with odd orders. For more references concerning partitions of graphs or hypergraphs, see [3,4,7,8,11,13,18–24].

Partitioning problems of directed graphs (digraphs for short) tend to be more difficult than that of undirected graphs. In contrast to undirected graphs, where Max-Cut only needs to optimize a single parameter (the total number of crossing edges), the analogous problem in digraphs is to measure the size of the directed cut in each direction. For a digraph D and $S, T \subseteq V(D)$, we use $e(S,T)$ to denote the number of arcs of D directed from S to T. In [21], Scott posed the following natural problem: What is the maximum constant c_d such that every digraph D with m arcs and minimum outdegree d admits a bipartition $V(D) = V_1 \cup V_2$ such that

$$\min\{e(V_1,V_2), e(V_2,V_1)\} \geq c_d m?$$

For $d = 1$, consider the graph $K_{1,n-1}$ and add a single edge inside the part of size $n-1$. This graph can be oriented easily so that the minimum outdegree is 1 and $\min\{e(V_1,V_2), e(V_2,V_1)\} \leq 1$ for every partition $V(D) = V_1 \cup V_2$. Hence, $c_1 = 0$. Lee, Loh and Sudakov [16] initiated the study of this problem and conjectured that

Conjecture 1.1 *(Lee, Loh and Sudakov [16]). Let d be an integer satisfying $d \geq 2$. Every digraph D with m arcs and minimum outdegree at least d admits a bipartition $V(D) = V_1 \cup V_2$ such that*

$$\min\{e(V_1,V_2), e(V_2,V_1)\} \geq \Big(\frac{d-1}{2(2d-1)} + o(1)\Big)m.$$

In the same paper, Lee, Loh and Sudakov verified Conjecture 1.1 for $d = 2, 3$, and noted that their method is not adequate for $d \geq 4$.

Even for oriented graphs, Conjecture 1.1 seems difficult to prove. It is not clear to us whether or not the minimum outdegree condition alone is sufficient for Conjecture 1.1 with large d. However, if we relax it to minimum degree or semidegree for oriented graphs, then we have the following results.

Theorem 1.2. *Every oriented graph D with m arcs and minimum degree $\delta \geq 1$ admits a bipartition $V(D) = V_1 \cup V_2$ such that*

$$\min\{e(V_1,V_2), e(V_2,V_1)\} \geq \Big(\frac{\delta-1}{4\delta} + o(1)\Big)m.$$

Theorem 1.3. *Let d be an integer with $d \geq 21$. Every oriented graph D with m arcs and minimum semidegree d admits a bipartition $V(D) = V_1 \cup V_2$ such that*

$$\min\{e(V_1,V_2), e(V_2,V_1)\} \geq \Big(\frac{d}{2(2d+1)} + o(1)\Big)m.$$

Please cite this article in press as: J. Hou, S. Wu, Bipartitions of oriented graphs, J. Combin. Theory Ser. B (2018), https://doi.org/10.1016/j.jctb.2018.03.003