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A unified treatment of linked and lean tree-decompositions

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ABSTRACT

There are many results asserting the existence of tree-decompositions of minimal width which still represent local connectivity properties of the underlying graph, perhaps the best known being Thomas' theorem that proves for every graph G the existence of a linked tree-decomposition of width $\mathrm{tw}(G)$. We prove a general theorem on the existence of linked and lean tree-decompositions, providing a unifying proof of many known results in the field, as well as implying some new results. In particular we prove that every matroid M admits a lean tree-decomposition of width $\mathrm{tw}(M)$, generalizing the result of Thomas.

1. Introduction

Given a tree T and vertices $t_1, t_2 \in V(T)$ let us denote by t_1Tt_2 the unique path in T between t_1 and t_2. Given a graph G a *tree-decomposition* of G is a pair $(T, \mathcal{V})$ consisting of a tree T, together with a collection of subsets of vertices $\mathcal{V} = \{V_t \subseteq V(G) : t \in V(T)\}$, called *bags*, such that:

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- $V(G) = \bigcup_{t \in T} V_t$;
- For every edge $e \in E(G)$ there is a t such that e lies in V_t;
- $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_2 \in V(t_1 T t_3)$.

The *width* of this tree-decomposition is the quantity $\max\{|V_t| - 1 : t \in V(T)\}$ and its *adhesion* is $\max\{|V_t \cap V_{t'}| : (t, t') \in E(T)\}$. Given a graph G its *tree-width* $\mathrm{tw}(G)$ is the smallest k such that G has a tree-decomposition of width k.

Definition 1.1. A tree decomposition $(T, \mathcal{V})$ of a graph G is called *linked* if for all $k \in \mathbb{N}$ and every $t, t' \in V(T)$, either G contains k disjoint V_t-$V_{t'}$ paths or there is an $s \in V(tTt')$ such that $|V_s| < k$.

Robertson and Seymour showed the following:

Theorem 1.2 *(Robertson and Seymour [17]). Every graph G admits a linked tree-decomposition of width $< 3 \cdot 2^{tw(G)}$.*

This result was an essential part of their proof that for any k the set of graphs with tree-width less than k is well-quasi-ordered by the minor relation. Thomas gave a new proof of Theorem 1.2, improving the bound on the tree-width of the linked tree-decomposition from $3 \cdot 2^{\mathrm{tw}(G)} - 1$ to the best possible value of $\mathrm{tw}(G)$.

Theorem 1.3 *(Thomas [20]). Every graph G admits a linked tree-decomposition of width $tw(G)$.*

In fact he showed a stronger result.

Definition 1.4. A tree decomposition $(T, \mathcal{V})$ of a graph G is called *lean* if for all $k \in \mathbb{N}$, $t, t' \in V(T)$ and vertex sets $Z_1 \subseteq V_t$ and $Z_2 \subseteq V_{t'}$ with $|Z_1| = |Z_2| = k$, either G contains k disjoint $Z_1 - Z_2$ paths or there exists an edge $\{s, s'\} \in E(tTt')$ with $|V_s \cap V_{s'}| < k$.

Thomas showed that every graph has a lean tree-decomposition of width $\mathrm{tw}(G)$. It is relatively simple to make a lean tree-decomposition linked without increasing its width: we subdivide each edge $(s, s') \in E(T)$ by a new vertex u and add $V_u := V_s \cap V_{s'}$. The real strength of Definition 1.4 in comparison to Definition 1.1 is the case $t = t'$. Broadly, Definition 1.1 tells us that the 'branches' in the tree-decomposition are no larger than the connectivity of the graph requires, if the separators $V_s \cap V_{s'}$ along a path in T are large, then G is highly connected along this branch. The case $t = t'$ of Definition 1.4 tells us that the bags are also no larger than their 'external connectivity' in G requires.

Bellenbaum and Diestel [2] used some of the ideas from Thomas' paper to give short proofs of two known results concerning tree-decompositions, the first Theorem 1.3 and the other the tree-width duality theorem of Seymour and Thomas [19]. Very similar ideas appear in the literature in multiple proofs of the existence of 'linked' or 'lean'

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